Exercise Sheet – Mathematical Analysis III

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练习 1. 设 $f(x)$ 是 $[a,b]$ 上的连续函数, 证明: $\forall x \in (a,b)$,

$$\int_a^x \mathrm{d}x_1 \int_a^{x_1} \mathrm{d}x_2 \cdots \int_a^{x_n} f(x_{n+1})\,\mathrm{d}x_{n+1} = \frac{1}{n!}\int_a^x (x-t)^n f(t)\,\mathrm{d}t.$$

解答 1. (数学归纳法) $n=1$ 时, 只需要将左端累次积分换顺后即可得到结果. 假设对 $n=k-1$ 时结论成立, 即

$$\int_a^x \mathrm{d}x_1 \int_a^{x_1} \mathrm{d}x_2 \cdots \int_a^{x_{k-1}} f(x_k)\,\mathrm{d}x_k = \frac{1}{(k-1)!}\int_a^x (x-t)^{k-1} f(t)\,\mathrm{d}t.$$

下面说明对 $n=k$ 时结论也成立, 事实上

$$\begin{aligned}
&\int_a^x \mathrm{d}x_1 \int_a^{x_1} \mathrm{d}x_2 \cdots \int_a^{x_k} f(x_{k+1})\,\mathrm{d}x_{k+1}\\
=&\int_a^x \left[\int_a^{x_1} \mathrm{d}x_2 \cdots \int_a^{x_k} f(x_{k+1})\,\mathrm{d}x_{k+1}\right]\mathrm{d}x_1\\
=&\int_a^x \frac{1}{(k-1)!}\int_a^{x_1} (x_1-t)^{k-1} f(t)\,\mathrm{d}t\,\mathrm{d}x_1 \quad \text{(由归纳假设)}\\
=&\frac{1}{(k-1)!}\int_a^x \left[\int_t^x (x_1-t)^{k-1}\,\mathrm{d}x_1\right] f(t)\,\mathrm{d}t \quad \text{(换序)}\\
=&\frac{1}{(k-1)!}\int_a^x \frac{(x-t)^k}{k} f(t)\,\mathrm{d}t\\
=&\frac{1}{k!}\int_a^x (x-t)^k f(t)\,\mathrm{d}t.
\end{aligned}$$

练习 2. 设 $f_i(x)$, $i=1,2,\ldots,n$ 以及 $g_j(x)$, $j=1,2,\ldots,n$ 是 $[a,b]$ 上的可积函数, 证明:

$$\frac{1}{n!}\int_{[a,b]^n} \det[f_i(x_j)]_{i,j=1}^n \det[g_i(x_j)]_{i,j=1}^n \,\mathrm{d}x_1 \ldots \mathrm{d}x_n = \det\left[\int_a^b f_i(x) g_j(x)\,\mathrm{d}x\right]_{i,j=1}^n.$$

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解答 2. 定义矩阵 $M = [M_{ij}]_{i,j=1}^n$ 为:

$$M_{ij} = \int_a^b f_i(x) g_j(x) dx, \quad i, j = 1, \dots, n.$$

我们将证明以下恒等式:

$$\frac{1}{n!} \int_{[a,b]^n} \det[f_i(x_j)] \cdot \det[g_i(x_j)] dx_1 \dots dx_n = \det(M).$$

对于固定的 $x_1, \dots, x_n \in [a, b]$, 我们有:

$$\det[f_i(x_j)]_{i,j=1}^n = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n f_i(x_{\sigma(i)}),$$

$$\det[g_i(x_j)]_{i,j=1}^n = \sum_{\tau \in S_n} \operatorname{sgn}(\tau) \prod_{i=1}^n g_i(x_{\tau(i)}).$$

因此,

$$\det[f_i(x_j)]_{i,j=1}^n \cdot \det[g_i(x_j)]_{i,j=1}^n = \sum_{\sigma, \tau \in S_n} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau) \prod_{i=1}^n f_i(x_{\sigma(i)}) g_i(x_{\tau(i)}).$$

令

$$I = \int_{[a,b]^n} \det[f_i(x_j)] \cdot \det[g_i(x_j)] dx_1 \dots dx_n.$$

代入展开式:

$$I = \sum_{\sigma, \tau \in S_n} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau) \int_{[a,b]^n} \prod_{i=1}^n f_i(x_{\sigma(i)}) g_i(x_{\tau(i)}) dx_1 \dots dx_n.$$

对于固定的 σ, τ, 作变量替换:

$$y_i = x_{\sigma(i)}, \quad i = 1, \dots, n.$$

则 $dx_1 \dots dx_n = dy_1 \dots dy_n$, 且

$$x_{\tau(i)} = y_{\sigma^{-1}(\tau(i))}.$$

令 $\rho = \sigma^{-1}\tau$, 则 $\operatorname{sgn}(\sigma) \operatorname{sgn}(\tau) = \operatorname{sgn}(\rho)$, 且积分变为:

$$\int_{[a,b]^n} \prod_{i=1}^n f_i(y_i) g_i(y_{\rho(i)}) dy_1 \dots dy_n =: J(\rho).$$

因此,

$$I = \sum_{\sigma \in S_n} \sum_{\rho \in S_n} \operatorname{sgn}(\rho) J(\rho) = n! \sum_{\rho \in S_n} \operatorname{sgn}(\rho) J(\rho).$$

现在计算 $J(\rho)$. 我们有:

$$J(\rho) = \int_{[a,b]^n} \prod_{i=1}^n f_i(y_i) g_i(y_{\rho(i)}) dy_1 \dots dy_n.$$

在第二个乘积中, 令 $j=\rho(i)$, 则 $i=\rho^{-1}(j)$, 所以:

$$\prod_{i=1}^{n} g_i(y_{\rho(i)}) = \prod_{j=1}^{n} g_{\rho^{-1}(j)}(y_j).$$

因此,

$$J(\rho) = \int_{[a,b]^n} \left(\prod_{i=1}^{n} f_i(y_i)\right) \left(\prod_{j=1}^{n} g_{\rho^{-1}(j)}(y_j)\right) dy_1 \dots dy_n.$$

由于积分区域是立方体且被积函数是乘积形式, 我们可以分离变量:

$$J(\rho) = \prod_{j=1}^{n} \int_a^b f_j(y) g_{\rho^{-1}(j)}(y) dy = \prod_{j=1}^{n} M_{j,\rho^{-1}(j)}.$$

令 $\sigma = \rho^{-1}$, 则 $\operatorname{sgn}(\rho) = \operatorname{sgn}(\sigma)$, 且

$$\prod_{j=1}^{n} M_{j,\rho^{-1}(j)} = \prod_{j=1}^{n} M_{j,\sigma(j)}.$$

因此,

$$\sum_{\rho \in S_n} \operatorname{sgn}(\rho) J(\rho) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} M_{j,\sigma(j)} = \det(M).$$

代入得:

$$I = n! \cdot \det(M) \quad \Rightarrow \quad \frac{1}{n!} I = \det(M).$$

至此, 我们已经证明了:

$$\frac{1}{n!} \int_{[a,b]^n} \det[f_i(x_j)] \cdot \det[g_i(x_j)] dx_1 \dots dx_n = \det \left[\int_a^b f_i(x) g_j(x) dx \right]_{i,j=1}^{n}.$$

练习 3. 假设 f 是 $[0,1]$ 上的可积函数, 证明:

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin^3(y) f(\sin(x)\sin(y)) dx dy = \frac{\pi}{4} \int_0^1 (t^2+1) f(t) dt.$$

解答 3. 令

$$I = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin^3(y) f(\sin(x)\sin(y))\, dx\, dy.$$

固定 y, 作变量替换 $u = \sin x$. 则 $du = \cos x\, dx = \sqrt{1-u^2}\, dx$, 所以 $dx = \frac{du}{\sqrt{1-u^2}}$. 当 $x=0$ 时, $u=0$; 当 $x=\frac{\pi}{2}$ 时, $u=1$. 因此

$$\int_0^{\frac{\pi}{2}} f(\sin x \sin y)\, dx = \int_0^1 \frac{f(u \sin y)}{\sqrt{1-u^2}}\, du.$$

代回原式:

$$I=\int_0^{\frac{\pi}{2}}\sin^3(y)\left[\int_0^1\frac{f(u\sin y)}{\sqrt{1-u^2}}\,du\right]dy.$$

交换积分次序:

$$I=\int_0^1\frac{1}{\sqrt{1-u^2}}\left[\int_0^{\frac{\pi}{2}}\sin^3(y)f(u\sin y)\,dy\right]du.$$

现在, 固定 u, 作变量替换 $t=u\sin y$. 则 $dt=u\cos y\,dy=u\sqrt{1-\sin^2 y}\,dy=\sqrt{u^2-t^2}\,dy$, 所以

$$dy=\frac{dt}{\sqrt{u^2-t^2}}.$$

当 $y=0$ 时, $t=0$; 当 $y=\frac{\pi}{2}$ 时, $t=u$. 同时, $\sin^3 y=\left(\frac{t}{u}\right)^3$. 因此

$$\int_0^{\frac{\pi}{2}}\sin^3(y)f(u\sin y)\,dy=\int_0^u\frac{t^3}{u^3}f(t)\cdot\frac{1}{\sqrt{u^2-t^2}}\,dt=\frac{1}{u^3}\int_0^u\frac{t^3f(t)}{\sqrt{u^2-t^2}}\,dt.$$

代回原式:

$$I=\int_0^1\frac{1}{\sqrt{1-u^2}}\cdot\frac{1}{u^3}\left[\int_0^u\frac{t^3f(t)}{\sqrt{u^2-t^2}}\,dt\right]du.$$

再次交换积分次序:

$$I=\int_0^1 t^3f(t)\left[\int_{u=t}^1\frac{1}{u^3\sqrt{1-u^2}\sqrt{u^2-t^2}}\,du\right]dt.$$

定义

$$J(t)=\int_{u=t}^1\frac{1}{u^3\sqrt{1-u^2}\sqrt{u^2-t^2}}\,du,$$

于是

$$I=\int_0^1 t^3f(t)J(t)\,dt.$$

现在计算 $J(t)$. 令 $u=\frac{t}{\sqrt{1-v^2}}$. 则

$$u^2=\frac{t^2}{1-v^2},\quad u^2-t^2=\frac{t^2v^2}{1-v^2},\quad \sqrt{u^2-t^2}=\frac{tv}{\sqrt{1-v^2}}.$$

同时,

$$du=\frac{tv}{(1-v^2)^{3/2}}\,dv,\quad u^3=\frac{t^3}{(1-v^2)^{3/2}},$$

且

$$\sqrt{1-u^2}=\sqrt{1-\frac{t^2}{1-v^2}}=\frac{\sqrt{1-v^2-t^2}}{\sqrt{1-v^2}}.$$

代入被积函数:

$$\frac{1}{u^3\sqrt{1-u^2}\sqrt{u^2-t^2}}\,du=\frac{1-v^2}{t^3\sqrt{1-v^2-t^2}}\,dv.$$

当 $u=t$ 时, $v=0$; 当 $u=1$ 时, $v=\sqrt{1-t^2}$. 因此

$$J(t)=\frac{1}{t^3}\int_0^{\sqrt{1-t^2}}\frac{1-v^2}{\sqrt{1-v^2-t^2}}\,dv.$$

现在令 $w=\frac{v}{\sqrt{1-t^2}}$, 于是 $v=w\sqrt{1-t^2}$, $dv=\sqrt{1-t^2}\,dw$. 当 $v=0$ 时, $w=0$; 当 $v=\sqrt{1-t^2}$ 时, $w=1$. 同时,

$$1-v^2=1-(1-t^2)w^2,\quad \sqrt{1-v^2-t^2}=\sqrt{1-t^2}\cdot\sqrt{1-w^2}.$$

代入得:

$$J(t)=\frac{1}{t^3}\int_0^1\frac{1-(1-t^2)w^2}{\sqrt{1-w^2}}\,dw.$$

即

$$J(t)=\frac{1}{t^3}\left[\int_0^1\frac{1}{\sqrt{1-w^2}}\,dw-(1-t^2)\int_0^1\frac{w^2}{\sqrt{1-w^2}}\,dw\right].$$

计算:

$$\int_0^1\frac{1}{\sqrt{1-w^2}}\,dw=\frac{\pi}{2},\quad \int_0^1\frac{w^2}{\sqrt{1-w^2}}\,dw=\frac{\pi}{4},$$

所以

$$J(t)=\frac{1}{t^3}\left[\frac{\pi}{2}-(1-t^2)\cdot\frac{\pi}{4}\right]=\frac{1}{t^3}\cdot\frac{\pi}{4}(1+t^2)=\frac{\pi}{4}\cdot\frac{1+t^2}{t^3}.$$

代回 I 的表达式:

$$I=\int_0^1 t^3f(t)\cdot\frac{\pi}{4}\cdot\frac{1+t^2}{t^3}\,dt=\frac{\pi}{4}\int_0^1(1+t^2)f(t)\,dt.$$

证明完成.

练习 4. 证明:

$$\int_0^1\int_0^1(xy)^{xy}\,\mathrm{d}x\,\mathrm{d}y=\int_0^1 y^y\,\mathrm{d}y.$$

解答 4. 令 $u=xy,\ v=y$. 则 $x=\frac{u}{v}$, 雅可比行列式为

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix} y & x \\ 0 & 1\end{vmatrix}=y,$$

故 $dx\,dy=\frac{1}{v}\,du\,dv$. 映射后积分区域为 $\{(u,v):0<v<1,\ 0<u<v\}$, 于是

$$\begin{aligned}\int_0^1\int_0^1(xy)^{xy}\,dx\,dy&=\int_{v=0}^1\int_{u=0}^v u^u\frac{1}{v}\,du\,dv=\int_0^1\frac{1}{v}\Big(\int_0^v u^u\,du\Big)dv\\&=\int_0^1 u^u\Big(\int_{v=u}^1\frac{1}{v}\,dv\Big)du=\int_0^1 u^u(-\ln u)\,du.\end{aligned}$$

由于 $\dfrac{d}{du}u^u = u^u(\ln u + 1)$ 且 $u^u \to 1$ 当 $u \to 0^+$，得

$$\int_0^1 u^u(\ln u + 1)\, du = \left[u^u\right]_0^1 = 1 - 1 = 0,$$

即 $\displaystyle\int_0^1 u^u\, du = -\int_0^1 u^u \ln u\, du$. 故

$$\int_0^1 \int_0^1 (xy)^{xy}\, dx\, dy = \int_0^1 u^u\, du.$$

练习 5 (高斯积分)**.**

(1) 计算积分:

$$\int_{\mathbb{R}^n} e^{-\sum_{i=1}^n x_i^2}\, \mathrm{d}x_1 \ldots \mathrm{d}x_n.$$

(2) 设 A 是一个 $n \times n$ 正定实对称矩阵, 则高斯积分公式为:

$$\int_{\mathbb{R}^n} \exp\left(-\langle Ax, x\rangle\right) dx = \frac{\pi^{n/2}}{\sqrt{\det A}}.$$

(3) 设 A 为 n 阶正定矩阵, $\alpha \in \mathbb{R}^n$, 证明:

$$\int_{\mathbb{R}^n} \exp\left(-\langle Ax, x\rangle + \langle \alpha, x\rangle\right) \mathrm{d}x = \frac{\pi^{n/2}}{\sqrt{\det A}} \exp\left(\frac{\langle A^{-1}\alpha, \alpha\rangle}{4}\right).$$

解答 5. (1) 这是书上 244 页的第 7 题.

(2) 由于 A 是实对称正定矩阵, 存在正交矩阵 Q 和对角矩阵 $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, 其中 $\lambda_i > 0$, 使得:

$$A = Q\Lambda Q^T$$

作变量替换 $x = Qy$, 且由于 Q 是正交矩阵, 雅可比行列式为 $|\det Q| = 1$, 因此 $dx = dy$. 原积分变为:

$$\int_{\mathbb{R}^n} \exp\left(-\langle Ax, x\rangle\right) dx = \int_{\mathbb{R}^n} \exp\left(-\langle Q\Lambda Q^T x, x\rangle\right) dx$$

利用内积在正交变换下的不变性:

$$\langle Q\Lambda Q^T x, x\rangle = \langle \Lambda Q^T x, Q^T x\rangle = \langle \Lambda y, y\rangle = \sum_{i=1}^n \lambda_i y_i^2$$

因此:

$$\int_{\mathbb{R}^n} \exp\left(-\langle Ax, x\rangle\right) dx = \int_{\mathbb{R}^n} \exp\left(-\sum_{i=1}^n \lambda_i y_i^2\right) dy.$$

由于被积函数可以分离变量:

$$\exp\left(-\sum_{i=1}^{n}\lambda_i y_i^2\right)=\prod_{i=1}^{n}\exp(-\lambda_i y_i^2),$$

且积分区域是 $\mathbb{R}^n$, 我们可以将多重积分写为一维积分的乘积:

$$\int_{\mathbb{R}^n}\exp\left(-\sum_{i=1}^{n}\lambda_i y_i^2\right)dy=\prod_{i=1}^{n}\int_{-\infty}^{\infty}\exp(-\lambda_i y_i^2)dy_i$$

利用标准的一维高斯积分公式:

$$\int_{-\infty}^{\infty}\exp(-at^2)dt=\sqrt{\frac{\pi}{a}},\quad a>0$$

对于每个 i, 我们有:

$$\int_{-\infty}^{\infty}\exp(-\lambda_i y_i^2)dy_i=\sqrt{\frac{\pi}{\lambda_i}}$$

因此:

$$\prod_{i=1}^{n}\int_{-\infty}^{\infty}\exp(-\lambda_i y_i^2)dy_i=\prod_{i=1}^{n}\sqrt{\frac{\pi}{\lambda_i}}=\pi^{n/2}\prod_{i=1}^{n}\lambda_i^{-1/2}.$$

注意到:

$$\prod_{i=1}^{n}\lambda_i=\det\Lambda=\det A.$$

因为:

$$\det A=\det(Q\Lambda Q^T)=\det Q\cdot\det\Lambda\cdot\det Q^T=\det\Lambda=\prod_{i=1}^{n}\lambda_i.$$

所以:

$$\prod_{i=1}^{n}\lambda_i^{-1/2}=\left(\prod_{i=1}^{n}\lambda_i\right)^{-1/2}=(\det A)^{-1/2}.$$

最终得到:

$$\int_{\mathbb{R}^n}\exp\left(-\langle Ax,x\rangle\right)dx=\frac{\pi^{n/2}}{\sqrt{\det A}}.$$

(3) 考虑二次型:

$$\langle Ax,x\rangle-\langle\alpha,x\rangle.$$

我们将其写为完全平方形式, 令

$$b=\frac{1}{2}A^{-1}\alpha,$$

则

$$\langle Ax,x\rangle-\langle\alpha,x\rangle=\langle A(x-b),x-b\rangle-\langle Ab,b\rangle.$$

(验证:

$$\langle A(x-b), x-b\rangle = \langle Ax, x\rangle - 2\langle Ax, b\rangle + \langle Ab, b\rangle,$$

而

$$\langle \alpha, x\rangle = 2\langle Ab, x\rangle,$$

因此

$$\langle Ax, x\rangle - \langle \alpha, x\rangle = \langle A(x-b), x-b\rangle - \langle Ab, b\rangle.$$

)

计算常数项:

$$\langle Ab, b\rangle = \left\langle A\left(\frac{1}{2}A^{-1}\alpha\right), \frac{1}{2}A^{-1}\alpha\right\rangle = \frac{1}{4}\langle \alpha, A^{-1}\alpha\rangle.$$

所以

$$\langle Ax, x\rangle - \langle \alpha, x\rangle = \langle A(x-b), x-b\rangle - \frac{1}{4}\langle \alpha, A^{-1}\alpha\rangle.$$

代入原积分:

$$I = \exp\left(\frac{1}{4}\langle \alpha, A^{-1}\alpha\rangle\right)\int_{\mathbb{R}^n}\exp\left(-\langle A(x-b), x-b\rangle\right)dx.$$

作变量替换 $y_j = x_j - b_j$, 则 $dx_j = dy_j$, 且积分区域仍为 $\mathbb{R}^n$. 因此

$$I = \exp\left(\frac{1}{4}\langle \alpha, A^{-1}\alpha\rangle\right)\int_{\mathbb{R}^n}\exp\left(-\langle Ay, y\rangle\right)dy.$$

利用高斯积分公式:

$$\int_{\mathbb{R}^n}\exp\left(-\langle Ay, y\rangle\right)dy = \frac{\pi^{n/2}}{\sqrt{\det A}},$$

于是有所求结果.

练习 6. 计算:

$$I = \int_0^\infty\int_0^\infty e^{-x-y}\frac{\cos(2k\sqrt{xy})}{\sqrt{xy}}dxdy.$$

解答 6. 令:

$$x = r^2\cos^2\theta, \quad y = r^2\sin^2\theta,$$

其中 $r \geq 0$, $0 \leq \theta \leq \frac{\pi}{2}$. 可以知道: $\sqrt{xy} = r^2\cos\theta\sin\theta$, $dxdy = 4r^3\cos\theta\sin\theta drd\theta$, $e^{-x-y} = e^{-r^2}$, $\cos(2k\sqrt{xy}) = \cos(kr^2\sin 2\theta)$. 代入积分:

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}\frac{\cos(kr^2\sin 2\theta)}{r^2\cos\theta\sin\theta}\cdot 4r^3\cos\theta\sin\theta drd\theta \\ &= 4\int_0^{\frac{\pi}{2}}\int_0^\infty re^{-r^2}\cos(kr^2\sin 2\theta)drd\theta. \end{aligned}$$

令 $u=r^2$, 则 $du=2rdr$, 内层积分变为

$$\int_0^\infty re^{-r^2}\cos(kr^2\sin 2\theta)dr=\frac{1}{2}\int_0^\infty e^{-u}\cos(\alpha u)du,$$

其中 $\alpha=k\sin 2\theta$.

利用积分公式

$$\int_0^\infty e^{-u}\cos(\alpha u)du=\frac{1}{1+\alpha^2},$$

得到

$$\int_0^\infty re^{-r^2}\cos(kr^2\sin 2\theta)dr=\frac{1}{2(1+k^2\sin^2 2\theta)}.$$

代入外层积分, 我们有:

$$I=4\int_0^{\frac{\pi}{2}}\frac{1}{2(1+k^2\sin^2 2\theta)}d\theta=2\int_0^{\frac{\pi}{2}}\frac{d\theta}{1+k^2\sin^2 2\theta}.$$

令 $\varphi=2\theta$, 则 $d\theta=\frac{d\varphi}{2}$, 积分限变为 0 到 π. 因此

$$I=2\int_0^{\pi}\frac{1}{1+k^2\sin^2\varphi}\cdot\frac{d\varphi}{2}=\int_0^{\pi}\frac{d\varphi}{1+k^2\sin^2\varphi}.$$

利用标准积分公式:

$$\int_0^{\pi}\frac{d\varphi}{1+a\sin^2\varphi}=\frac{\pi}{\sqrt{1+a}},$$

取 $a=k^2$, 得

$$I=\frac{\pi}{\sqrt{1+k^2}}.$$