Exercise Sheet – Mathematical Analysis III

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Exercise 1 (Warm up: Directional derivatives and gradient)**.** Complete the following exercises.

(1) Let $f(x, y, z) = x^2y + yz^3$. Compute the gradient ∇f at the point $(1, 2, 1)$.

(2) For $f(x, y, z)$ as above, compute the directional derivative of f at $(1, 2, 1)$ in the direction of the vector $\vec{v} = (2, -1, 2)$.

(3) Let $f(x, y) = x^3 - 3xy^2$. Find all points (x, y) where the gradient ∇f is parallel to the vector $(1, 1)$.

Application in ML:

(4) (Linear regression) A common loss function is Mean Squared Error (MSE). For a single data point (x, y), the loss is defined as $L(m, b) = (y - (mx + b))^2$, where m is the slope and b is the y-intercept of the regression line. Compute the gradient $\nabla L(m, b)$ and interpret its components in terms of how they influence the loss.

(5) (Gradient descent step) Suppose you are using gradient descent to minimize the function $J(\theta_0, \theta_1) = \theta_0^2 + 2\theta_1^2$. Calculate the gradient $\nabla J(\theta_0, \theta_1)$ firstly, and write down the update rule for θ_0 and θ_1 using a learning rate α.

Solution 2. (1) The gradient is

$$\nabla f(x, y, z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right).$$

Compute:

$$\frac{\partial f}{\partial x} = 2xy, \quad \frac{\partial f}{\partial y} = x^2 + z^3, \quad \frac{\partial f}{\partial z} = 3yz^2.$$

At $(1, 2, 1)$:

$$\frac{\partial f}{\partial x} = 2 \cdot 1 \cdot 2 = 4, \quad \frac{\partial f}{\partial y} = 1^2 + 1^3 = 2, \quad \frac{\partial f}{\partial z} = 3 \cdot 2 \cdot 1^2 = 6.$$

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So,

$$\nabla f(1,2,1) = (4,\, 2,\, 6).$$

(2) The directional derivative in direction $\vec{v} = (2, -1, 2)$ is

$$D_{\vec{v}} f = \nabla f \cdot \frac{\vec{v}}{|\vec{v}|}.$$

Compute $|\vec{v}| = \sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{9} = 3$. So, unit vector $\vec{u} = \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$. Then,

$$D_{\vec{v}} f = 4 \cdot \frac{2}{3} + 2 \cdot \left(-\frac{1}{3}\right) + 6 \cdot \frac{2}{3} = \frac{8}{3} - \frac{2}{3} + \frac{12}{3} = \frac{18}{3} = 6.$$

(3) Compute $\nabla f(x,y) = (3x^2 - 3y^2,\, -6xy)$. We want $\nabla f(x,y)$ parallel to $(1,1)$, i.e.,

$$(3x^2 - 3y^2,\, -6xy) = \lambda(1,1).$$

So,

$$3x^2 - 3y^2 = \lambda, \quad -6xy = \lambda.$$

Equate:

$$x^2 - y^2 + 2xy = 0 \implies (x+y)^2 - 2y^2 = 0.$$

So,

$$x + y = \pm\sqrt{2}y.$$

Thus, all points (x,y) with $x = (\sqrt{2} - 1)y$ or $x = (-\sqrt{2} - 1)y$.

For $\nabla f(x,y)$ to be parallel to $(1,1)$, there must exist $\lambda \in \mathbb{R}$ such that

$$(3x^2 - 3y^2,\, -6xy) = \lambda(1,1).$$

So,

$$3x^2 - 3y^2 = \lambda, \quad -6xy = \lambda.$$

Equate the two expressions for λ:

$$3x^2 - 3y^2 = -6xy \implies x^2 + 2xy - y^2 = 0.$$

This factors as

$$(x+y)^2 - 2y^2 = 0 \implies (x+y)^2 = 2y^2 \implies x + y = \pm\sqrt{2}\, y.$$

Thus,

$$x = (\sqrt{2} - 1)y \quad \text{or} \quad x = (-\sqrt{2} - 1)y.$$

So all points (x,y) with $x = (\sqrt{2} - 1)y$ or $x = (-\sqrt{2} - 1)y$ have ∇f parallel to $(1,1)$.

(4)
$$\frac{\partial L}{\partial m} = -2x(y-(mx+b))$$
$$\frac{\partial L}{\partial b} = -2(y-(mx+b))$$

So,
$$\nabla L(m,b) = \big(-2x(y-(mx+b)),\ -2(y-(mx+b))\big)$$

Interpretation: the gradient points in the direction of steepest increase of the loss. The m-component shows how changing the slope affects the loss (scaled by x), and the b-component shows how changing the intercept affects the loss. Both are proportional to the residual $y-(mx+b)$.

(5) The gradient is
$$\nabla J(\theta_0,\theta_1) = \left(\frac{\partial J}{\partial \theta_0}, \frac{\partial J}{\partial \theta_1}\right) = (2\theta_0,\ 4\theta_1).$$

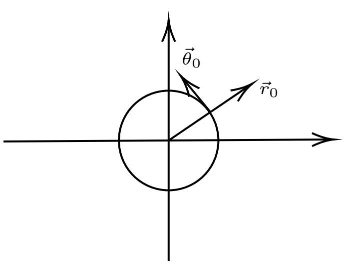
The gradient descent update rule is:
$$\theta_0^{\text{new}} = \theta_0 - \alpha \cdot 2\theta_0 = \theta_0(1-2\alpha)$$
$$\theta_1^{\text{new}} = \theta_1 - \alpha \cdot 4\theta_1 = \theta_1(1-4\alpha)$$

where α is the learning rate.

Exercise 3. Complete the following exercises

(1) Let $u=f(x,y)$, $x=r\cos\theta$, $y=r\sin\theta$. Prove that
$$\nabla u = \frac{\partial f}{\partial r}\vec{r}_0 + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{\theta}_0,$$

where $\vec{r}_0=(\cos\theta,\sin\theta)$ and $\vec{\theta}_0=(\cos(\theta+\frac{\pi}{2}),\cos\theta)$.



(2) We have the following equation:
$$\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1.$$

Prove that

$$(\nabla u)^2 = 2 < \vec{A}, \nabla u >,$$

where $\vec{A} := (x, y, z)$.
(Hint: see $u = u(x, y, z)$)

Solution 4. (1) Recall that $\nabla u = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ in Cartesian coordinates.

By the chain rule:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x},$$
$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y}.$$

Compute the partial derivatives:

$$r = \sqrt{x^2 + y^2} \implies \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta, \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta,$$

$$\theta = \arctan\left(\frac{y}{x}\right) \implies \frac{\partial \theta}{\partial x} = -\frac{y}{r^2} = -\frac{\sin\theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{x}{r^2} = \frac{\cos\theta}{r}.$$

Therefore,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\cos\theta - \frac{\partial f}{\partial \theta}\frac{\sin\theta}{r},$$
$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\sin\theta + \frac{\partial f}{\partial \theta}\frac{\cos\theta}{r}.$$

So

$$\nabla u = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \frac{\partial f}{\partial r}(\cos\theta, \sin\theta) + \frac{1}{r}\frac{\partial f}{\partial \theta}(-\sin\theta, \cos\theta).$$

Notice that $(\cos\theta, \sin\theta) = \vec{r}_0$ and $(-\sin\theta, \cos\theta) = (\cos(\theta + \frac{\pi}{2}), \sin(\theta + \frac{\pi}{2})) = \vec{\theta}_0$. Thus,

$$\nabla u = \frac{\partial f}{\partial r}\vec{r}_0 + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{\theta}_0.$$

(Vector decomposition)

$$\nabla u =< \nabla u, \vec{r}_0 > \vec{r}_0 + < \nabla u, \vec{\theta}_0 > \vec{\theta}_0.$$

Then you can directly calculate $< \nabla u, \vec{r}_0 >$ and $< \nabla u, \vec{\theta}_0 >$.

(2) Noticing that

$$(\nabla u)^2 = 2 < \vec{A}, \nabla u > \Leftrightarrow \; u_x^2 + u_y^2 + u_z^2 = 2(xu_x + yu_y + zu_z).$$

Differentiate

$$\frac{x^2}{a^2 + u} + \frac{y^2}{b^2 + u} + \frac{z^2}{c^2 + u} = 1$$

with respect to x:

$$\frac{2x(a^2+u)-u_x x^2}{(a^2+u)^2}-\frac{y^2u_x}{(b^2+u)^2}-\frac{z^2u_x}{(x^2+u)^2}=0,$$

that is,

$$\frac{2x}{a^2+u}=\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]u_x. \tag{1}$$

Using similar manners (rotation), we have

$$\frac{2y}{b^2+u}=\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]u_y, \tag{2}$$

$$\frac{2z}{c^2+u}=\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]u_z. \tag{3}$$

Multiply x (y, z) to the both sides of the above three equations respectively, and add them together (also notice the condition)

$$2=\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right](zu_x+yu_y+zu_z).$$

Square both sides of the mentioned above three equations (1)–(3), and add them together:

$$4=\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right](u_x^2+u_y^2+u_z^2).$$

Combined the above two equations, we arrive at the desired result.

Exercise 5. Suppose the function $f(x,y)$ has a nonzero directional derivative at some point (x_0,y_0), and the directional derivatives along three different (non-colinear) directions at (x_0,y_0) are equal. Prove that $f(x,y)$ is not differentiable at (x_0,y_0).

Solution 6. Suppose that $f(x,y)$ is differentiable at (x_0,y_0). Therefore there exist partial derivatives of $f(x,y)$ at (x_0,y_0):

$$A:=f'_x(x_0,y_0),\quad B:=f'_y(x_0,y_0),$$

and

$$df(x_0,y_0)=Adx+Bdy.$$

By the condition, there exist three different direction $\vec{l_i}$ $(i = 1, 2, 3)$ such that

$$\frac{\partial f}{\partial \vec{l_i}}(x_0, y_0) = A\cos\theta_i + B\sin\theta_i = C \neq 0.$$

We have the following linear systems:

$$\begin{aligned} A\cos\theta_1 + B\sin\theta_1 - C = 0, \\ A\cos\theta_2 + B\sin\theta_2 - C = 0, \\ A\cos\theta_3 + B\sin\theta_3 - C = 0. \end{aligned}$$

We will take advantage of the following truth:

[The linear equations only has trivial solution $(A = B = C = 0)$, if and only if

$$\det \begin{pmatrix} \cos\theta_1 & \sin\theta_1 & 1 \\ \cos\theta_2 & \sin\theta_2 & 1 \\ \cos\theta_2 & \sin\theta_2 & 1 \end{pmatrix} \neq 0.$$

]

Noting that $\exists \alpha, \beta \neq 0$ such that

$$\vec{l_1} = \alpha\vec{l_2} + \beta\vec{l_3} \implies \cos\theta_1 = \alpha\cos\theta_2 + \beta\cos\theta_3, \quad \sin\theta_1 = \alpha\sin\theta_2 + \beta\sin\theta_3.$$

Square both sides and add them together

$$1 = \alpha^2 + \beta^2 + 2\alpha\beta\cos(\theta_2 - \theta_3) \overset{(\theta_2 \neq \theta_3)}{\neq} (\alpha + \beta)^2 \implies \alpha + \beta \neq 1.$$

Then

$$\begin{aligned} \det \begin{pmatrix} \cos\theta_1 & \sin\theta_1 & 1 \\ \cos\theta_2 & \sin\theta_2 & 1 \\ \cos\theta_2 & \sin\theta_2 & 1 \end{pmatrix} &= \det \begin{pmatrix} \alpha\cos\theta_2 + \beta\cos\theta_3 & \alpha\sin\theta_2 + \beta\sin\theta_3 & 1 \\ \cos\theta_2 & \sin\theta_2 & 1 \\ \cos\theta_2 & \sin\theta_2 & 1 \end{pmatrix} \\ &= \det \begin{pmatrix} 0 & 0 & 1 \\ \cos\theta_2 & \sin\theta_2 & 1 - \alpha - \beta \\ \cos\theta_2 & \sin\theta_2 & 1 \end{pmatrix} = (1 - \alpha - \beta)\sin(\theta_3 - \theta_2) \neq 0. \end{aligned}$$

Thus $(A, B, C) = (0, 0, 0)$.