Transient asymptotics of the modified Camassa-Holm (mCH) equation

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Outline

- Introduction
- Main results
- Strategy of the proofs ($\bar{\partial}$ nonlinear steepest analysis to an RH problem)

Introduction

The mCH equation

The Cauchy problem for the modified Camassa-Holm (mCH) equation

$$\begin{aligned} & m_t + \left(m\left(u^2 - u_x^2\right)\right)_x = 0, \quad m = u - u_{xx}, \quad u = u(x,t), \qquad x \in \mathbb{R},\ t > 0, \\ & u(x,0) = u_0(x), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x \in \mathbb{R}, \end{aligned}$$

with nonzero boundary condition

$$u_0(x) \to \omega, \quad |x| \to \infty.$$

- Physical interpretation: mCH equation provides a model for the unidirectional propagation of shallow water waves of mildly large amplitude over a flat bottom, where $u(x,t)$ is interpreted as the horizontal velocity in certain level of fluid.
- $\omega > 0$ is a constant that is related to the critical shallow water wave speed. WLOG, we fix $\underline{\omega = 1}$.

Lax pair of the mCH equation:

$$\Phi_x = X\Phi, \qquad \Phi_t = T\Phi,$$

where

$$X = \frac{1}{2}\begin{pmatrix} -1 & \lambda m \\ -\lambda m & 1 \end{pmatrix}, \qquad \lambda = -\frac{1}{2}(z + z^{-1}),$$

$$T = \begin{pmatrix} \lambda^{-2} + \frac{1}{2}(u^2 - u_x^2) & -\lambda^{-1}(u - u_x + 1) - \frac{1}{2}\lambda(u^2 - u_x^2)m \\ \lambda^{-1}(u + u_x + 1) + \frac{1}{2}\lambda(u^2 - u_x^2)m & -\lambda^{-2} - \frac{1}{2}(u^2 - u_x^2) \end{pmatrix}.$$

[Schiff, '96; Qiao,'06]

$$\boxed{\text{The mCH equation} \iff X_t - T_x + [X, T] = 0}$$

↖ Compatible condition

Compatible condition for (Lax) integrable PDEs: [Lax,'68]

The mCH equation

History of the mCH equation:

- Traced back to the work of Fokas.

  [Fokas, '95]

- The mCH equation was introduced by applying the general method of tri-Hamiltonian duality to the bi-Hamiltonian representation of the modified KdV equation.

  [Fuchssteiner,'96; Olver-Rosenau,'96]

- Among the list of Novikov in the classification of generalized Camassa-Holm type equations.

  [Novikov,'09]

- Recognized as an integrable modification of the celebrated Camassa-Holm (CH) equation

$$m_t + (um)_x + u_x m = 0, \quad m = u - u_{xx}.$$

The mCH equation

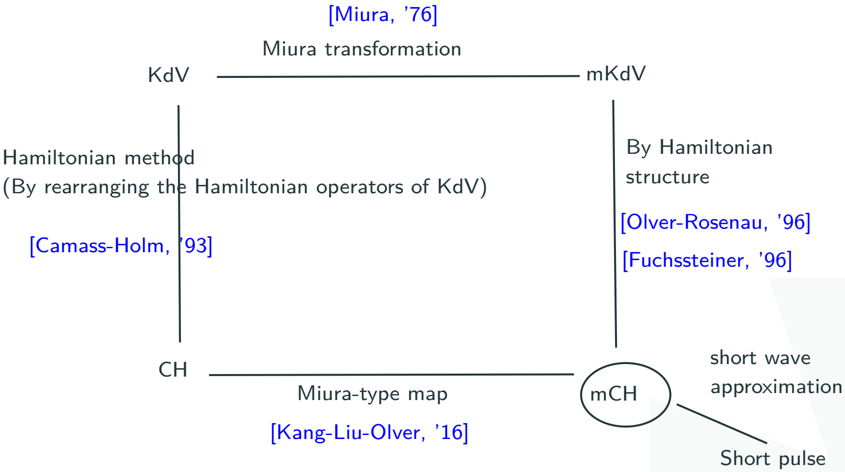

Figure 1: Connections to other well-known equations.

The mCH equation

Some results about the mCH equation:

- The wave-breaking, peakon solutions.

  [Gui-Liu-Olver-Qu,'13; Chang-Szmigielski, '18]

- The quasi-periodic algebro-geometric solutions.

  [Hou-Fan-Qiao,'17]

- The Bäcklund transformation and the related nonlinear superposition formulae.

  [Wang-Liu-Miao,'20]

- ...

An Riemann-Hilbert (RH) formalism of the Cauchy problem has recently been developed.

[Bouted de Monvel-Karpenko-Shepelsky,'20]

Long-time asymptotics for the mCH equation

An Riemann-Hilbert (RH) formalism of the Cauchy problem has recently been developed.

[Bouted de Monvel-Karpenko-Shepelsky,'20]

Long-time asymptotics of $u(x,t)$ shows qualitatively different behaviors in different regions of the (x,t)-half plane.

- a soliton region: $\{(x,t) : \xi > 2\}$,
- a fast decay region: $\{(x,t) : \xi < -1/4\}$,
- two oscillatory regions: $\{(x,t) : 0 < \xi < 2\} \cup \{(x,t) : -1/4 < \xi < 0\}$.

Here, ξ is the velocity denoted by $\xi := x/t$.

RK: Initial data belongs to Schwartz space (smooth, rapidly decreasing).

[Bouted de Monvel-Karpenko-Shepelsky,'22]

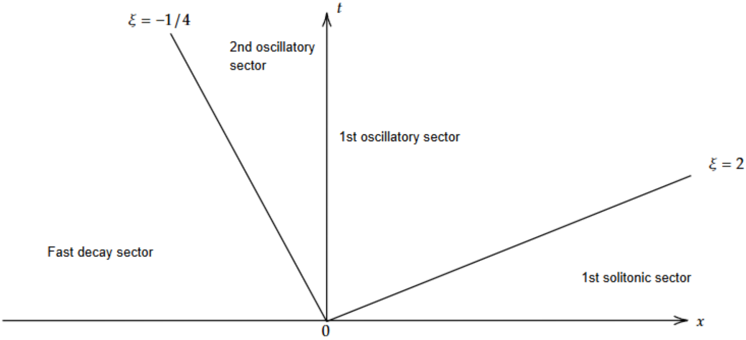

Figure 2: The different regular asymptotic regions of the (x, t)-half plane, where $\xi = x/t$.

Our topic

QUESTION:

What are the asymptotics when $\xi \approx -1/4$ and $\xi \approx 2$?

Our topic

Long time asymptotics of the mCH equation in three transition regions.

- The first transition region (Painlevé) $\mathcal{R}_I := \{(x,t) : 0 \leqslant |\xi - 2| t^{2/3} \leqslant C\}$,
- The second transition region (Painlevé) $\mathcal{R}_{II} := \{(x,t) : 0 \leqslant |\xi + 1/4| t^{2/3} \leqslant C\}$,
- The third transition region (collisionless shock) $\mathcal{R}_{III} := \{(x,t) : 2 \cdot 3^{1/3} (\log t)^{2/3} < (2-\xi) t^{2/3} < C(\log t)^{2/3},\ C > 2 \cdot 3^{1/3}\}$.

Our topic

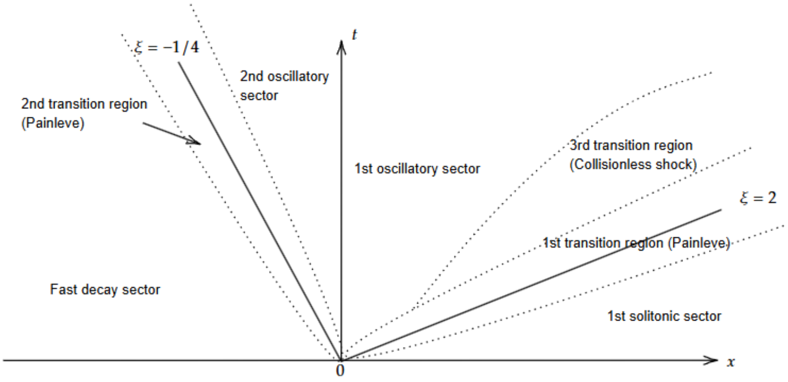

Figure 3: The different asymptotic regions of the (x, t)-half plane, where $\xi = x/t$.

Main results

Assumptions

We assume that the initial data satisfies the following conditions.

- $m_0(x) := m(x,0) = u_0(x) - u_{0xx}(x) > 0$ for $x \in \mathbb{R}$.
- $m_0(x) - 1 \in H^{2,1}(\mathbb{R}) \cap H^{1,2}(\mathbb{R})$, where
$$H^{k,s}(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) : \langle \cdot \rangle^s \partial_x^j f \in L^2(\mathbb{R}), j = 0, 1, \dots, k \right\}, \quad s \geqslant 0,$$
with $\langle x \rangle := (1 + x^2)^{1/2}$ is the weighted Sobolev space.

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Global solution of the Cauchy problem for mCH equation exists uniquely.

[Yang-Fan-Liu,'22]

Theorem (X.-Yang-Zhang, JLMS, '24)

Let $u(x,t)$ be the global solution of the Cauchy problem for the mCH equation over the real line under assumptions, and denote by $r(z)$ and $\{z_n\}_{n=1}^{2\mathcal{N}}$, $|z_n| = 1$, the reflection coefficient and the discrete spectrum associated to the initial data $u_0(x)$ in the lower half plane. As $t \to +\infty$, we have the following asymptotics of $u(x,t)$ in the above transient regions $\mathcal{R}_I - \mathcal{R}_{III}$ given above.

(a) *For $\xi \in \mathcal{R}_I$,*

$$u(x,t) = 1 - (2/81)^{-1/3} t^{-2/3} v'(s) + \mathcal{O}(t^{-\min\{1-4\delta_1, 1/3+9\delta_1\}}),$$

where δ_1 is any real number belonging to $(1/24, 1/18)$,

$$s = 6^{-2/3}\left(\frac{x}{t} - 2\right) t^{2/3},$$

and $v(s)$ is the unique solution of the Painlevé II equation

$$v''(s) = sv(s) + 2v^3(s)$$

characterized by

$$v(s) \sim r(1)\mathrm{Ai}(s), \qquad s \to +\infty,$$

with Ai *being the classical Airy function and $r(1) \in [-1, 1]$.*

Transient asymptotics of the mCH equation

Theorem (X.-Yang-Zhang, JLMS, '24)

(b) For $\xi \in \mathcal{R}_{II}$,

$$u(x,t) = 1 + 3^{-2/3} t^{-1/3} f_{II}(s) v_{II}(s) + \mathcal{O}\left(t^{\max\{-2/3+4\delta_2, -1/3-5\delta_2\}}\right),$$

where δ_2 *is any real number belonging to* $(0, 1/15)$, $s = -\left(\frac{8}{9}\right)^{1/3}\left(\frac{x}{t} + \frac{1}{4}\right) t^{2/3}$,

$$\begin{aligned} f_{II}(s) = {} & 2\sqrt{2+\sqrt{3}}\left(\sin\psi_a(s,t)\cos\gamma_a - \frac{iT_1}{T(i)}\cos\psi_a(s,t)\sin\gamma_a\right) \\ & + 2\sqrt{2-\sqrt{3}}\left(\sin\psi_b(s,t)\cos\gamma_b - \frac{iT_1}{T(i)}\cos\psi_b(s,t)\sin\gamma_b\right) \\ & + \sqrt{3}\cos\left(\frac{\Lambda_a+\Lambda_b}{2}\right)\sin\left(\frac{\Lambda_a+\Lambda_b}{2}\right) \end{aligned}$$

with $\gamma_a = \arctan(2+\sqrt{3})$, $\gamma_b = \arctan(2-\sqrt{3})$,

$$\psi_a(s,t) = \frac{3^{7/6}}{2} s t^{1/3} + \frac{3\sqrt{3}}{4} t + \Lambda_a, \qquad \psi_b(s,t) = -\frac{3^{7/6}}{2} s t^{1/3} - \frac{3\sqrt{3}}{4} t + \Lambda_b,$$

$$\begin{aligned} \Lambda_a = {} & \arg r(2+\sqrt{3}) + 4\sum_{n=1}^{\mathcal{N}} \arg(2+\sqrt{3}-z_n) \\ & - \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\log(1-|r(\zeta)|^2)}{\zeta-(2+\sqrt{3})}\,\mathrm{d}\zeta - 2\sqrt{3}\log(T(i)), \end{aligned}$$

and $v_{II}(s)$ *is the unique solution of the Painlevé II equation characterized by* $v_{II}(s) \sim -|r(2+\sqrt{3})|\mathrm{Ai}(s)$ *as* $s \to +\infty$ *with* $|r(2+\sqrt{3})| < 1$.

Theorem (X.-Yang-Zhang, JLMS, '24)

(c) For $\xi \in \mathcal{R}_{III}$, if $|r(\pm 1)| = 1$, $r \in H^s$ with $s > 5/2$, we have

$$u(x,t) = 1 - \frac{(2-\xi)(a-b)q}{12p} \cdot$$

$$\left(-i \cdot \frac{\Theta\left(A(\infty) - \frac{\varkappa}{4}\right)}{\Theta\left(A(\infty) - \frac{\varkappa}{4} + \frac{\phi}{\pi}\right)} \cdot \left(\frac{\Theta\left(-A(\hat{k}) - \frac{\varkappa}{4} + \frac{\phi}{\pi}\right)}{\Theta\left(-A(\hat{k}) - \frac{\varkappa}{4}\right)} \right)' \Big|_{\hat{k}=\infty} e^{i\phi} \right.$$

$$\left. + \frac{\Theta\left(A(\infty) - \frac{\varkappa}{4}\right) \Theta\left(-A(\infty) - \frac{\varkappa}{4} + \frac{\phi}{\pi}\right)}{\Theta\left(-A(\infty) - \frac{\varkappa}{4}\right) \Theta\left(A(\infty) - \frac{\varkappa}{4} + \frac{\phi}{\pi}\right)} e^{i\phi} \right) (1 + o(1)),$$

where p and q are two fixed positive constants, $\Theta(z)$ is the Jacobi theta function defined by

$$\Theta(s) := \sum_{n \in \mathbb{Z}} e^{2\pi i n s + \varkappa \pi i n^2},$$

and

$$A(\hat{k}) = \left(2 \int_b^a \frac{1}{w_+(\zeta)} \mathrm{d}\zeta \right)^{-1} \int_b^{\hat{k}} \frac{1}{w(\zeta)} \mathrm{d}\zeta, \qquad \hat{k} \in \mathbb{C} \setminus [-b, b],$$

with $w^2(\hat{k}) = (\hat{k}^2 - a^2)(\hat{k}^2 - b^2)$ is the Abel integral.

Strategy of the proofs

An RH charactrization of the mCH equation

The Cauchy problem of the mCH equation is related to the following RH problem.

- $M^{(1)}(z)$ is meromorphic for $z \in \mathbb{C} \setminus \mathbb{R}$ with simple poles in the set $\mathcal{Z} \cup \mathcal{Z}^*$.
- For $z \in \mathbb{R}$, we have

$$M_+^{(1)}(z) = M_-^{(1)}(z) V^{(1)}(z)$$

  where

$$V^{(1)}(z) = \begin{pmatrix} 1 - |r(z)|^2 & r(z)e^{2i\theta(z)} \\ -\bar{r}(z)e^{-2i\theta(z)} & 1 \end{pmatrix}$$

  with

$$\theta(z) := \theta(z; y, t) = -\frac{t}{4}\left(z - \frac{1}{z}\right)\left(\hat{\xi} - \frac{8}{(z + 1/z)^2}\right), \ \hat{\xi} := \frac{y}{t}.$$

  and where

$$y(x) = x - \int_x^{+\infty} (m(\zeta) - 1)\, \mathrm{d}\zeta.$$

  is a space scaling variable.
- For each $z_j \in \mathcal{Z}$, $j = 1, \dots, 2\mathcal{N}$, we have the following residue conditions:

$$\operatorname*{Res}_{z=z_j} M(z) = \lim_{z \to z_j} M(z) \begin{pmatrix} 0 & c_j e^{2i\theta(z_j)} \\ 0 & 0 \end{pmatrix},$$

$$\operatorname*{Res}_{z=\bar{z}_j} M(z) = \lim_{z \to \bar{z}_j} M(z) \begin{pmatrix} 0 & 0 \\ \bar{c}_j e^{-2i\theta(\bar{z}_j)} & 0 \end{pmatrix},$$

  where c_j is the norming constant associated with z_j.
- As $z \to \infty$ in $\mathbb{C} \setminus \mathbb{R}$, we have $M^{(1)}(z) = I + \mathcal{O}(z^{-1})$.

An RH charactrization of the mCH equation

Suppose the initial data satisfies our assumptions, the scattering data $\left(r, \{z_j, c_j\}_{j=1}^{2\mathcal{N}}\right)$ belongs to $\left(H^{1,2}(\mathbb{R}) \cap H^{2,1}(\mathbb{R})\right) \otimes \mathbb{C}^{2\mathcal{N}} \otimes \mathbb{C}^{2\mathcal{N}}$, and the above RH problem admits a unique solution. Moreover, we can use the local behaviors of RH problem at $z = 0$ and $z = i$ to characterize the solution for the Cauchy problem of the mCH equation in the following way.

[Yang-Fan-Liu,'22]

Let $M^{(1)}(z;y,t)$ is the unique solution of above RH problem, we have

$$M^{(1)}(0;y,t)=\begin{pmatrix}\alpha(y,t) & i\beta(y,t)\\ -i\beta(y,t) & \alpha(y,t)\end{pmatrix},$$

where $\alpha(y,t)$, $\beta(y,t)$ are real functions satisfying $\alpha^2-\beta^2=1$. If $\beta\neq 0$, we have

$$\begin{aligned}M^{(1)}(z;y,t)=&\begin{pmatrix}f_1(y,t) & \frac{i\beta}{\alpha+1}f_2(y,t)\\ -\frac{i\beta}{\alpha+1}f_1(y,t) & f_2(y,t)\end{pmatrix}\\ &+\begin{pmatrix}\frac{i\beta}{\alpha+1}g_1(y,t) & g_2(y,t)\\ g_1(y,t) & -\frac{i\beta}{\alpha+1}g_2(y,t)\end{pmatrix}(z-i)+\mathcal{O}\left((z-i)^2\right),\quad z\to i,\end{aligned}$$

where $g_1(y,t)$, $g_2(y,t)$, $f_1(y,t)$, $f_2(y,t)$ are real functions.

The solution $u(x,t)=u(x(y,t),t)$ of the Cauchy problem can then be expressed in the following parametric form:

$$x(y,t)=y+2\log\left(\alpha_1(y,t)\right),$$

$$u(y,t)=1-\alpha_2(y,t)\alpha_1(y,t)-\alpha_3(y,t)\alpha_1(y,t)^{-1},$$

where

$$\alpha_1(y,t)=\left(1-\frac{\beta}{\alpha+1}\right)f_1,\quad \alpha_2(y,t)=\frac{\beta}{\alpha+1}f_2+\left(1-\frac{\beta}{\alpha+1}\right)g_2,$$

$$\alpha_3(y,t)=\frac{-\beta}{\alpha+1}f_1+\left(1-\frac{\beta}{\alpha+1}\right)g_1.$$

From $M^{(1)}$ to a holomorphic RH problem

Residue conditions $\rightsquigarrow$ Jump conditions on the auxiliary contours. After a suitable transformation, we have an RH problem for $M^{(2)}$.

Jump conditions for $M^{(2)}$:

$$M_+^{(2)}(z) = M_-^{(2)}(z) V^{(2)}(z),$$

where

$$V^{(2)}(z) = \begin{cases} \begin{pmatrix} 1 & e^{2i\theta(z)} r(z) T^{-2}(z) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -e^{-2i\theta(z)} \bar{r}(z) T^{2}(z) & 1 \end{pmatrix}, & z \in \mathbb{R} \setminus \mathrm{I}(\hat{\xi}), \\ \begin{pmatrix} 1 & 0 \\ -\frac{e^{-2i\theta(z)} \bar{r}(z) T_-^{2}(z)}{1-|r(z)|^2} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{e^{2i\theta(z)} r(z) T_+^{-2}(z)}{1-|r(z)|^2} \\ 0 & 1 \end{pmatrix}, & z \in \mathrm{I}(\hat{\xi}), \\ \begin{pmatrix} 1 & 0 \\ -c_n^{-1}(z - z_n) e^{-2i\theta(z_n)} T^{2}(z) & 1 \end{pmatrix}, & z \in \partial\mathbb{D}_n, \\ \begin{pmatrix} 1 & -\bar{c}_n^{-1}(z - \bar{z}_n) e^{2i\theta(\bar{z}_n)} T^{-2}(z) \\ 0 & 1 \end{pmatrix}, & z \in \partial\mathbb{D}_n^*, \end{cases}$$

From $M^{(1)}$ to a holomorphic RH problem

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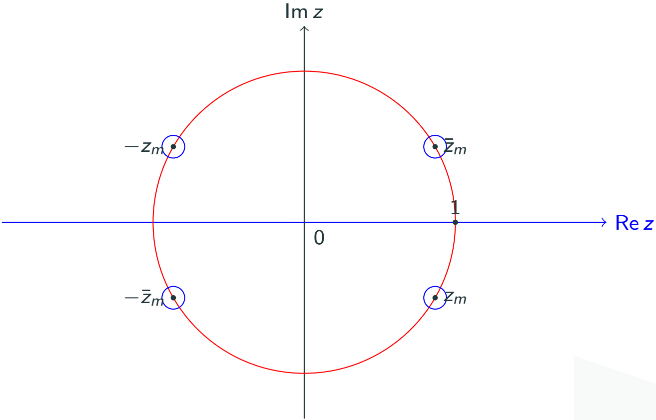

Figure 4: The red circle represents the unit circle. The blue circles around the poles together with real axis are the boundaries of $\Sigma^{(2)}$.

From $M^{(1)}$ to a holomorphic RH problem

By the signature tables of $\operatorname{Im}\theta$, it is readily seen that $V^{(2)}(z) \to I$ as $t \to +\infty$ for $z \in \cup_{n=1}^{2\mathcal{N}}(\partial\mathbb{D}_n \cup \partial\mathbb{D}_n^*)$ exponentially fast. Thus, RH problem for $M^{(2)}$ is asymptotically equivalent to an RH problem for $M^{(3)}$ with an error bound $\mathcal{O}(e^{-ct})$ for some constant $c > 0$.

$\bar{\partial}$ nonlinear steepest descent analysis

Let the RH problem for $M^{(3)}$ be as the starting of performing $\bar{\partial}$ nonlinear steepest descent analysis.

[McLaughlin-Miller,'06] [Dieng-McLaughlin-Miller,'08]

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- Opening $\bar{\partial}$ lenses to construct a mixed $\bar{\partial}$-RH problem for $M^{(4)}$ (Since $r(z)$ is not an analytical function !).

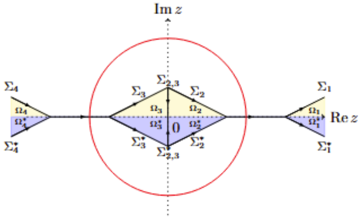

Figure 5: The jump contours of the RH problem for $M^{(4)}$.

$\bar\partial$ nonlinear steepest descent analysis

- $\bar\partial$-extension for the reflection coefficient:

$$d_1(z) := \left[\bar{R}(\mathrm{Re}\, z) - \bar{R}(k_1) - \bar{R}'(k_1)\,\mathrm{Re}(z-k_1)\right]\cos\left(\frac{\pi \arg(z-k_1)\,\mathcal{X}(\arg(z-k_1))}{2\varphi_0}\right) + \bar{R}(k_1) + \bar{R}'(k_1)(z-k_1)$$

with boundary condition

$$d_j(z) = \begin{cases} \bar{R}(z), & z \in \mathbb{R}, \\ \bar{R}(k_j) + \bar{R}'(k_j)^2(z-k_j), & z \in \Sigma_j. \end{cases}$$

- For each $j = 1, \ldots, 4$ and $z \in \Omega_j$, we have

$$|d_j(z)| \lesssim \sin^2\left(\frac{\pi}{2\varphi_0}\arg(z-k_j)\right) + \left(1 + \mathrm{Re}(z)^2\right)^{-1/2},$$

$$|\bar\partial d_j(z)| \lesssim |\mathrm{Re}\, z - k_j|^{1/2},$$

$$|\bar\partial d_j(z)| \lesssim |\mathrm{Re}\, z - k_j|^{-1/2} + \sin\left(\frac{\pi}{2\varphi_0}\arg(z-k_j)\,\mathcal{X}(\arg(z-k_j))\right),$$

$$|\bar\partial d_j(z)| \lesssim 1.$$

$\bar{\partial}$ nonlinear steepest descent analysis

Mixed $\bar{\partial}$-RH problem for $M^{(4)}$.

- $M^{(4)}(z)$ is continuous for $z \in \mathbb{C} \setminus \Sigma^{(4)}$.
- For $z \in \Sigma^{(4)}$, we have $M_+^{(4)}(z) = M_-^{(4)}(z) V^{(4)}(z)$, where

$$V^{(4)}(z) = \begin{cases} \begin{pmatrix} 1 & e^{2i\theta(z)} R(z) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -e^{-2i\theta(z)} \bar{R}(z) & 1 \end{pmatrix}, & z \in (k_4, k_3) \cup (k_2, k_1), \\ R^{(3)}(z)^{-1}, & z \in \Sigma_j, \ j = 1, 2, 3, 4, \\ R^{(3)}(z), & z \in \Sigma_j^*, \ j = 1, 2, 3, 4, \\ \begin{pmatrix} 1 & 0 \\ (d_2(z) - d_3(z)) e^{-2i\theta(z)} & 1 \end{pmatrix}, & z \in \Sigma_{2,3}, \\ \begin{pmatrix} 1 & (d_3^*(z) - d_2^*(z)) e^{2i\theta(z)} \\ 0 & 1 \end{pmatrix}, & z \in \Sigma_{2,3}^*. \end{cases}$$

- As $z \to \infty$ in $\mathbb{C} \setminus \Sigma^{(4)}$, we have $M^{(4)}(z) = I + \mathcal{O}(z^{-1})$.
- For $z \in \mathbb{C}$, we have the $\bar{\partial}$-derivative relation $\bar{\partial} M^{(4)}(z) = M^{(4)}(z) \bar{\partial} R^{(3)}(z)$, where

$$\bar{\partial} R^{(3)}(z) = \begin{cases} \begin{pmatrix} 0 & 0 \\ \bar{\partial} d_j(z) e^{-2i\theta(z)} & 0 \end{pmatrix}, & z \in \Omega_j, \ j = 1, ..., 4, \\ \begin{pmatrix} 0 & \bar{\partial} d_j^*(z) e^{2i\theta(z)} \\ 0 & 0 \end{pmatrix}, & z \in \Omega_j^*, \ j = 1, ..., 4, \\ 0, & \text{elsewhere}. \end{cases}$$

$\bar{\partial}$ nonlinear steepest descent analysis

Let the RH problem for $M^{(3)}$ be as the starting of performing $\bar{\partial}$ nonlinear steepest descent analysis.

[McLaughlin-Miller,'06] [Dieng-McLaughlin-Miller,'19]

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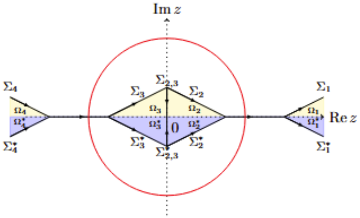


Figure 6: The jump contours of the RH problem for $M^{(4)}$.

- Decomposition into a pure RH problem for $N(z)$ (omitting $\bar{\partial}$-derivative part) and a pure $\bar{\partial}$-problem for $M^{(5)}(z)$ ($\bar{\partial}R^{(3)} \neq 0$).
- Construction of global and local parametrices for pure RH problem.
- Analysis to the $\bar{\partial}$-component.

The Painlevé region.

- The Painlevé II parametrix plays an important role in the analysis of the pure RH problem.

The collisionless shock region.

- Introduction of the g-function mechanism.
- A model RH problem solvable in terms of the Jacobi theta function.

Define

$$M^{(5)}(z) = M^{(4)}(z)N(z)^{-1}.$$

- $M^{(5)}(z)$ is continuous and has sectionally continuous first partial derivatives in $\mathbb{C}$.
- As $z \to \infty$ in $\mathbb{C}$, we have $M^{(5)}(z) = I + \mathcal{O}(z^{-1})$.
- The $\bar{\partial}$-derivative of $M^{(5)}$ satisfies $\bar{\partial}M^{(5)}(z) = M^{(5)}(z)W^{(3)}(z)$, $z \in \mathbb{C}$ with $W^{(3)}(z) = N(z)\bar{\partial}R^{(3)}(z)N(z)^{-1}$.

Solution of pure $\bar{\partial}$-problem can be expressed in terms of the integral equation

$$M^{(5)}(z) = I + \frac{1}{\pi}\iint_{\mathbb{C}} \frac{M^{(5)}(\zeta)W^{(3)}(\zeta)}{\zeta - z}\,\mathrm{d}\mu(\zeta),$$

where $\mu(\zeta)$ stands for the Lebesgue measure on $\mathbb{C}$. Introducing the left Cauchy-Green integral operator

$$f\mathcal{C}_z(z) = \frac{1}{\pi}\iint_{\mathbb{C}} \frac{f(\zeta)W^{(3)}(\zeta)}{\zeta - z}\,\mathrm{d}\mu(\zeta),$$

we could rewrite in an operator form

$$M^{(5)}(z) = I \cdot (I - \mathcal{C}_z)^{-1}.$$

Aim: Evaluate the norm of the integral operator $(I - \mathcal{C}_z)^{-1}$ so that we can estimate $M^{(5)}(z)$ as $t \to +\infty$.

Proof of our main theorems

- Recalling the series of transformations to obtain the asymptotics of $u(y,t)$ and $x(y,t)$ by using the reconstruction formulae in the RH problem for $M^{(1)}$.
- From (y,t)-half plane to (x,t)-half plane ($u(y,t) \rightsquigarrow u(x,t)$): properties of Painlevé II transcendent and Jacobi theta function.

Why Painlevé ?

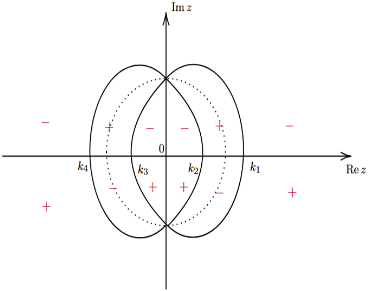
In the 1st transition region, the phase function $\theta(z)$ has four saddle points

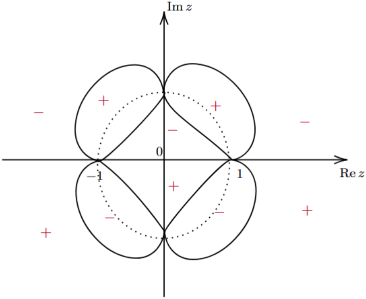
$$k_1 = 2\sqrt{s_+} + \sqrt{4s_+ + 1}, \qquad k_2 = -2\sqrt{s_+} + \sqrt{4s_+ + 1},$$
$$k_3 = 2\sqrt{s_+} - \sqrt{4s_+ + 1}, \qquad k_4 = -2\sqrt{s_+} - \sqrt{4s_+ + 1},$$

where

$$s_+ := \frac{1}{4\hat{\xi}}\left(-\hat{\xi} - 1 + \sqrt{1 + 4\hat{\xi}}\right).$$

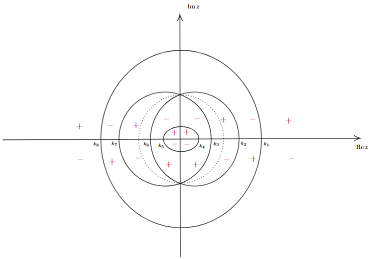
Why Painlevé ?

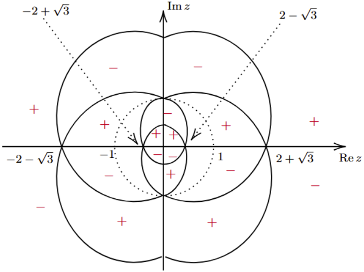




As $\hat{\xi} \to 2^-$, $k_{1,2} \to 1$, $k_{3,4} \to -1$. (demo)

Why Painlevé ?





As $\hat{\xi} \to \left(-\frac{1}{4}\right)^+$, $k_{1,2} \to 2+\sqrt{3}$, $k_{3,4} \to 2-\sqrt{3}$, $k_{5,6} \to -2+\sqrt{3}$, $k_{7,8} \to -2-\sqrt{3}$.
(demo)

Why does the collisionless shock region occur ?

About collisionless shock region:

- A terminology due to Gurevich and Pitaevsk

  [Gurevich-Pitaevski '74]

- First rigorous result for KdV equation.

  [Deift-Venakides-Zhou '94]

- It turns out the local RH problem near each k_j is controlled in the norm by $(1-|r(k_j)|^2)^{-1}$. In the generic case, i.e., $|r(\pm 1)| \equiv 1$, these norms blow up as $k_{1,2} \to 1$, $k_{3,4} \to -1$.

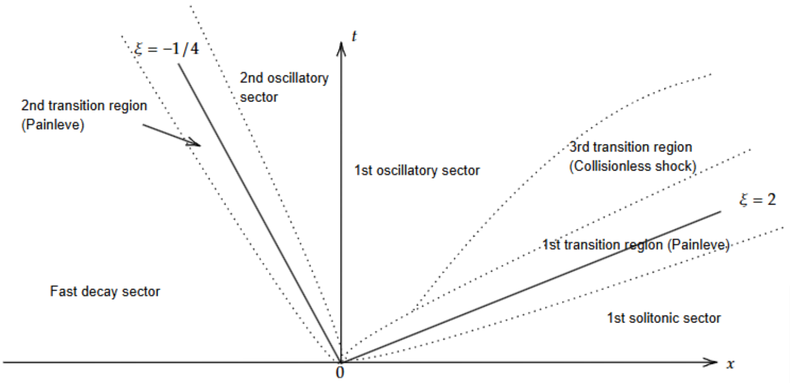


Figure 7: What happens near $\xi \approx 0$?

Near $\xi \approx 0$, we see from a rough calculation that the transition should occur in the sub-sub-leading term or higher order term of the large-time asymptotics.

Thanks for your attention !

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Comments and Questions are welcome !