Confluent hypergeometric kernel determinant on multiple large intervals

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Outline

Introduction

Unitary ensembles with Fisher–Hartwig (FH) singularity

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$$f_n(\lambda_1, \ldots, \lambda_n) = \frac{1}{\mathcal{Z}_n} \prod_{k=1}^{n} e^{-\lambda_k^2} \prod_{1 \leqslant k < j \leqslant n} (\lambda_k - \lambda_j)^2.$$

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- jump-type singularity & root-type singularity $\rightsquigarrow$ **Fisher-Hartwig** singularity (at $z = 0$).

Unitary ensembles with Fisher–Hartwig (FH) singularity

The jpdf of Dyson circular unitary ensemble with FH singularity:

$$f_n(\theta_1, \ldots, \theta_n) = \frac{1}{\mathcal{Z}'_n} \prod_{k=1}^{n} |1 - e^{i\theta_k}|^{2\alpha} e^{i\beta(\theta_k - \pi)} \prod_{1 \leqslant k < j \leqslant n} |e^{i\theta_k} - e^{i\theta_j}|^2,$$

where

- $\mathcal{Z}'_n$: normalization constant.
- $\theta_k \in [0, 2\pi)$, $k = 1, \ldots, n$.
- FH singularity at $z = 1$, (i.e., $\theta = 0$).

The correlation kernel

The above eigenvalues form a **determinant point process** (DPP), which is characterized by a correlation kernel $K_n(x, y; \alpha, \beta)$, that is,

$$\rho(\lambda_1, \ldots, \lambda_n) = \det \left[K_n(\lambda_i, \lambda_j; \alpha, \beta) \right]_{i,j=1}^{n}.$$

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- GUE: $K_n(x, y; \alpha, \beta) = \sqrt{w(x)}\sqrt{w(y)} \sum_{j=0}^{n-1} p_j(x) p_j(y)$.
  $p_j(x)$ are orthonormal polynomials with the weight function $w(x)$ over $\mathbb{R}$, i.e.,

$$\int_{\mathbb{R}} p_i(x) p_j(x) w(x) \, \mathrm{d}x = \delta_{ij}, \quad w(x) = e^{-x^2} |x|^{2\alpha} \chi_\beta(x).$$

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- CUE: $K_n(e^{i\theta}, e^{i\phi}; \alpha, \beta) = \sqrt{w(e^{i\theta})}\sqrt{w(e^{i\phi})}\sum_{j=0}^{n-1} p_j(e^{i\theta})\overline{p_j(e^{i\phi})}$.

$$\int_0^{2\pi} p_i(e^{i\theta})\overline{p_j(e^{i\theta})}w(e^{i\theta})\,\mathrm{d}\theta = \delta_{jk}, \quad w(z) = z^n|z-1|^{2\alpha}z^\beta e^{-i\pi\beta},\ z = e^{i\theta}.$$

Large-n limiting kernel

In the bulk (near the FH singular point), the correlation kernel $K_n(x, y; \alpha, \beta)$ converges to **confluent hypergeometric kernel** $K^{(\alpha,\beta)}(x, y)$ under a suitable scaling.

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- Unitary random matrix ensembles generated by a concrete weight function on the unit circle.

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- it describes the local statistics of eigenvalues in the bulk of the spectrum near a FH singular point for a broad class of unitary ensemble of random matrices.

The confluent hypergeometric kernel

The confluent hypergeometric (CH) kernel with two parameters $\alpha > -1/2$ and $\beta \in i\mathbb{R}$ is defined by

$$K^{(\alpha,\beta)}(x,y) = \frac{1}{2\pi i}\frac{\Gamma(1+\alpha+\beta)\Gamma(1+\alpha-\beta)}{\Gamma(1+2\alpha)^2}\frac{A(x)B(y)-A(y)B(x)}{x-y},$$

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- $\Gamma(z)$ denotes the usual Gamma function. $A(x)$ and $B(x)$ are defined by

$$A(x) := \chi_\beta(x)^{\frac{1}{2}} |2x|^\alpha e^{-ix} \phi(1+\alpha+\beta, 1+2\alpha; 2ix),$$
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- The confluent hypergeometric function is defined by

$$\phi(a,b;z) = \sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}\frac{z^n}{n!}, \qquad b \neq 0, -1, -2, \dots,$$

where $(z)_n := z(z+1)\cdots(z+n-1) = \frac{\Gamma(z+n)}{\Gamma(z)}$ is the Pochhammer symbol.

The confluent hypergeometric kernel arises in several different, but related areas

Universal features of the confluent hypergeometric kernel

The confluent hypergeometric kernel arises in several different, but related areas

- Infinite random matrices and Hua-Pickrell measure.

  [Borodin-Olshanski, '01]

- Representation theory.

  [Borodin-Deift, '01]

- Circular unitary ensemble.

  [Deift-Krasovsky-Vasilevska, '11]

Relations to other kernels

Confluent hypergeometric kernel can reduce to other kernels

- If $\beta = 0$, $\alpha \neq 0$, following from the relation

$$\phi(\alpha, 2\alpha; 2ix) = \Gamma\left(\alpha + \frac{1}{2}\right) e^{ix} \left(\frac{x}{2}\right)^{-\alpha+\frac{1}{2}} J_{\alpha-\frac{1}{2}}(x),$$

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  then we have type-I Bessel kernel

$$K^{(\alpha,0)}(x,y) \equiv K^{(\mathrm{Bessel1})}(x,y) =$$
$$\left(\frac{|x|}{x}\right)^{\alpha} \left(\frac{|y|}{y}\right)^{\alpha} \frac{\sqrt{xy}}{2} \frac{J_{\alpha+\frac{1}{2}}(x)J_{\alpha-\frac{1}{2}}(x) - J_{\alpha+\frac{1}{2}}(y)J_{\alpha-\frac{1}{2}}(x)}{x-y}.$$

[Akemann-Damgaard-Magnea-Nishigaki, '97] [Kuijlaars-Vanlessen, '03]

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An episode: Type-II Bessel kernel:

$$K^{(\mathrm{Bessel2})}(x,y) := \frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y}) - \sqrt{x}J'_\alpha(\sqrt{x})J_\alpha(\sqrt{y})}{2(x-y)}.$$

[Forrester, '93]

Relations to other kernels

- If $\alpha = 0$, $\beta \neq 0$, we have a degenerated confluent hypergeometric kernel.

  [Tibboel, '10] [Moreno- Martínez-Finkelshtein -Sousa, '11]

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- If $\alpha = \beta = 0$, we obtain the sine kernel

$$K^{(0,0)}(x,y) \equiv K^{(\mathrm{sine})}(x,y) = \frac{\sin(x-y)}{\pi(x-y)}.$$

Gap probability for DPP

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$$\mathbb{P}(\text{no points lie on } \Sigma) = \mathcal{F}(\Sigma) := \det\left(1 - \mathcal{K}|_{\Sigma}\right),$$

where $\mathcal{K}$ is an integration operator acting on $L^2(\Sigma)$ with integrable kernel $K(x, y)$.

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- Our interest: **large gap asymptotics**

$$\mathcal{F}(s\Sigma) := \det\left(1 - \mathcal{K}|_{s\Sigma}\right) = ?, \quad \text{as} \quad s \to +\infty,$$

especially the case that Σ is **a union of disjoint intervals**.

History: sine kernel determinant

$$\log\det\left(1-\mathcal{K}^{(\mathrm{sine})}|_{s\Sigma}\right) = C_1 s^2 + C_2 \log s + C_3 \log\theta(V(s)) + C_4 + \mathcal{O}\left(s^{-1}\right), \quad s\to+\infty.$$

[1] Combined the earlier work [Widom '71] on Toeplitz determinant.

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<tr><td>$(-1,1)$</td><td>[Dyson, '62]
[Cloizeaux-Mehta, '73]
[Widom, '94]
$C_1=-1/2$</td><td colspan="2">[Deift-Its-Zhou, '97]
$C_2=-1/4,\ C_3=0$</td><td>[Dyson, '76][1]
[Krasovsky, '04]
[Ehrhardt, '04]
[Deift-Its-Krasovsky, '07]
$C_4=(\log 2)/12+3\zeta'(-1)$</td></tr>
<tr><td>$(-1,v_1)\cup(v_2,1)$</td><td colspan="4">[Fahs-Krasovsky, '22]</td></tr>
<tr><td>$\cup_{j=0}^{n}(a_j,b_j)$</td><td colspan="3">[Deift-Its-Zhou, '97]
C_2: integral expression</td><td>?</td></tr>
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Table: Large gap asymptotics for the sine kernel determinant.

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History: Airy kernel determinant

$$\log\det\left(1-\mathcal{K}^{(\mathrm{Ai})}|_{s\Sigma}\right) = C_1 s^3 + C_2 \log s + C_3 \log\theta(V(s)) + C_4 + e(s), \quad s\to+\infty.$$

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$C_1=-1/12,\ C_2=-1/8,$
$C_3=0,\ e(s)=\mathcal{O}(s^{-3/2})$</td><td>[Deift-Its-Krasovsky, '08]
[Baik-Buckingham-DiFranco, '08]
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History: type-II Bessel kernel determinant

$$\log\det(1-\mathcal{K}^{(\mathrm{Bes2})}|_{s\Sigma}) = C_1 s + C_2 s^{1/2} + C_3 \log s + C_4 \log\theta(V(s)) + C_5 + \mathcal{O}(s^{-1/2}), \quad s \to +\infty.$$

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$(0,x_1)$	[Tracy-Widom, '94] $C_1=-x_1/4,\ C_2=\alpha\sqrt{x_1},$ $C_3=-\alpha^2/4,\ C_4=0$				$C_5=G(1+\alpha)(2\pi)^{-\alpha/2}-(\alpha^2\log x_1)/4$ [Ehrhardt, '10] $\rightsquigarrow\alpha\in(-1,1)$ [Deift-Krasovsky-Vasilevska, '11] $\rightsquigarrow\alpha\in(-1,+\infty)$
$\bigcup_{j=0}^{2g}(x_j,x_{j+1})$ $x_0=0$	[Blackstone-Charlier-Lenells, 21'] C_3: exact value (under the ergodic condition)				?

Table: Large gap asymptotics for the type-II Bessel kernel determinant.

History: Confluent hypergeometric kernel determinant

As $s \to +\infty$,

$$\log\det(1 - \mathcal{K}^{(\alpha,\beta)}|_{s\Sigma}) = C_1 s^2 + C_2 s + (\beta^2 - \alpha^2 + C_3)\log s + C_4 \log\theta(V(s)) + C_5 + \mathcal{O}(s^{-1}).$$

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$\cup_{j=0}^{n}(a_j,b_j)$ $0\in(a_m,b_m)$ for some m: $1\leqslant m\leqslant n$	?				

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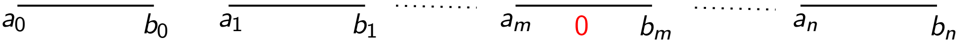
Today's topic

Aim: establish large gap asymptotics for the confluent hypergeometric kernel determinant $\det(1-\mathcal{K}^{(\alpha,\beta)})$ on multiple large intervals.



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$a_0 \quad b_0 \quad a_1 \quad b_1 \quad \cdots \quad a_m \quad \color{red}{0} \quad b_m \quad \cdots \quad a_n \quad b_n$

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$0\in(a_m,b_m)$ for some m: $1\leqslant m\leqslant n$</td><td colspan="4">[Xu-Zhang-Zhao, '25]</td><td>?</td></tr>
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Table: Large gap asymptotics for the confluent hypergeometric kernel determinant.

Main results

The Riemann surface

We will encounter a hyperelliptic Riemann surface $\mathcal{W}$ associated to the algebraic equation

$$\sqrt{\mathcal{R}(z)} := \sqrt{\prod_{j=0}^{n} (z-a_j)\,(z-b_j)}.$$

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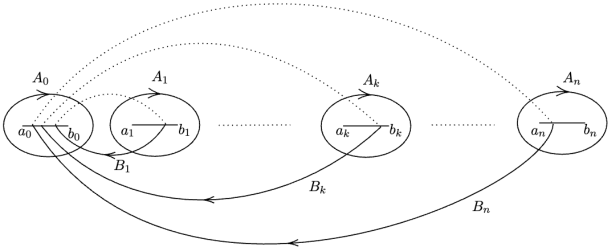

Figure: The canonical homology basis $\{A_j, B_j\}_{j=1}^{n}$ for the Riemann surface $\mathcal{W}$.

- $\sqrt{\mathcal{R}(z)} \sim \pm z^{n+1}$, as $z \to \infty$ on the first (second) sheet.
- Canonical homology basis $\{A_j, B_j\}$.

The $\mathbb{A}$-matrix:

$$\mathbb{A} := (a_{k,l})_{0\leqslant k\leqslant n, 0\leqslant l\leqslant n} = \begin{pmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,n} \\ a_{1,0} & a_{1,1} & \cdots & a_{1,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,0} & a_{n,1} & \cdots & a_{n,n} \end{pmatrix},$$

$$\tilde{\mathbb{A}} := (a_{k,l})_{k=1,\dots,n}^{l=0,\dots,n-1} = \begin{pmatrix} a_{1,0} & a_{1,1} & \cdots & a_{1,n-1} \\ a_{2,0} & a_{2,1} & \cdots & a_{2,n-1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,0} & a_{n,1} & \cdots & a_{n,n-1} \end{pmatrix},$$

$$\vec{a} = (a_{0,n+1}, a_{1,n+1}, \dots, a_{n,n+1})^{\mathrm{T}},$$

where $a_{k,l} := \oint_{A_k} \frac{z^l}{\sqrt{\mathcal{R}(z)}}\,\mathrm{d}z = 2i(-1)^{n-k+1} \int_{a_k}^{b_k} \frac{z^l}{|\mathcal{R}(z)|^{\frac{1}{2}}}\,\mathrm{d}z.$

Preliminaries – The $\mathbb{A}$-matrix

The $\mathbb{A}$-matrix:

$$\mathbb{A} := (a_{k,l})_{0\leqslant k\leqslant n, 0\leqslant l\leqslant n} = \begin{pmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,n} \\ a_{1,0} & a_{1,1} & \cdots & a_{1,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,0} & a_{n,1} & \cdots & a_{n,n} \end{pmatrix},$$

$$\tilde{\mathbb{A}} := (a_{k,l})_{k=1,\ldots,n}^{l=0,\ldots,n-1} = \begin{pmatrix} a_{1,0} & a_{1,1} & \cdots & a_{1,n-1} \\ a_{2,0} & a_{2,1} & \cdots & a_{2,n-1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,0} & a_{n,1} & \cdots & a_{n,n-1} \end{pmatrix},$$

$$\vec{a} = (a_{0,n+1}, a_{1,n+1}, \ldots, a_{n,n+1})^{\mathrm{T}},$$

where $a_{k,l} := \oint_{A_k} \frac{z^l}{\sqrt{\mathcal{R}(z)}}\,\mathrm{d}z = 2i(-1)^{n-k+1}\int_{a_k}^{b_k} \frac{z^l}{|\mathcal{R}(z)|^{\frac{1}{2}}}\,\mathrm{d}z.$

$\rightsquigarrow$ $\mathbb{A}$ and $\tilde{\mathbb{A}}$ are invertible.

[Farkas-Kra, '92, *Riemann Surfaces* 2nd ed]

Preliminaries – The basis of one-form

Introduce the basis of holomorphic one-forms:

$$\vec{\omega} := (\omega_1, \omega_2, \ldots, \omega_n) = \frac{\mathrm{d}z}{\sqrt{\mathcal{R}(z)}}(1, z, \ldots, z^{n-1})\tilde{\mathbb{A}}^{-1},$$

such that

$$\oint_{A_k} \omega_j = \delta_{jk}, \qquad j, k = 1, \ldots, n.$$

Meanwhile, we have the Riemann matrix of B_j periods:

$$\tau := (\tau_{ij})_{i,j=1}^{n} = \left(\oint_{B_j} \omega_i \right)_{i,j=1}^{n}.$$

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$\rightsquigarrow$ τ is symmetric and has a positively definite imaginary part, i.e, $-i\tau$ is positive definite.

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Preliminaries – Multi-dimensional Riemann-θ function and Abel's map

The multi-dimensional Riemann-θ function is defined by

$$\theta\left(\vec{z}\right) = \sum_{\vec{m}\in\mathbb{Z}^n} e^{2\pi i\vec{m}^{\mathrm{T}}\vec{z}+i\pi\vec{m}^{\mathrm{T}}\tau\vec{m}}, \qquad \vec{z} = \left(z_1, \ldots, z_n\right)^{\mathrm{T}} \in \mathbb{C}^n \text{ modulo } \mathbb{Z}^n.$$

- Converging absolutely and uniformly on compact sets of the $\mathbb{C}^n$.
- Even ($\theta(\vec{z}) = \theta(-\vec{z})$), entire function for $\vec{z} \in \mathbb{C}^n$.
- Periodic properties: $\theta(\vec{z} + \vec{e}_j) = \theta(\vec{z})$ and $\theta(\vec{z} \pm \vec{\tau}_j) = e^{\mp 2\pi i z_j - \pi i \tau_{jj}}\theta(\vec{z})$, where $\vec{e}_j = (0, \ldots, 1, \ldots, 0)^{\mathrm{T}}$ with 1 in the j-th position and $\vec{\tau} := \tau\vec{e}_j$.
- Vanishing at each odd half-period: $\theta(\vec{z}) = 0$ if $\vec{z} = \frac{\vec{m}}{2} + \frac{\tau\vec{n}}{2}$ with $\vec{m}, \vec{n} \in \mathbb{Z}^n$ and $\vec{m}^{\mathrm{T}}\vec{n}$ is odd.

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Abel's map:

$$\vec{\mathcal{A}}(z) := \int_{a_0}^{z} \vec{\omega}^{\mathrm{T}}.$$

Preliminaries – The polynomial, linear vector and frequencies

The following polynomial of degree $n+1$ plays an important role in our analysis:

$$\mathrm{p}(z) = z^{n+1} + \sum_{j=0}^{n} p_j z^j.$$

$\oint_{A_k} \frac{\mathrm{p}(s)}{\sqrt{\mathcal{R}(s)}} \,\mathrm{d}s = 0,\ k = 0, 1, \ldots, n. \implies (p_0, \ldots, p_n)^{\mathrm{T}} = -\mathbb{A}^{-1}\vec{a}.$

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A column vector with linear components:

$$\vec{V}(s) := (V_1(s), \ldots, V_n(s))^{\mathrm{T}}, \quad V_j(s) := \frac{s}{2\pi}\Omega_j + \frac{1}{2\pi}\operatorname{Im}(\zeta_j) \in \mathbb{R}.$$

- Frequencies $\Omega_j := 2\sum_{k=0}^{j-1}(-1)^{n-k}\int_{b_k}^{a_{k+1}} \frac{\mathrm{p}(s)}{|\mathcal{R}(s)|^{\frac{1}{2}}}\,\mathrm{d}s > 0.$
- $\operatorname{Im}\zeta_j \in \mathbb{R}$ is dependent of α and β.

Preliminaries – The linear flow on torus

A function $\mathcal{L} : \mathbb{C} \times \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{C}$ defined by

$$\mathcal{L}(z, \vec{\mu}) = \frac{h(z)}{\mathrm{p}(z)} \eta(z, \vec{\mu}).$$

where $h(z) = \prod_{k=0}^{n}(z - a_k) + \prod_{k=0}^{n}(z - b_k)$, and $\eta\,(z, \vec{\mu}) = \frac{\theta(\vec{0})^2 \theta(\vec{\mathcal{A}}(z)+\vec{\mu}+\vec{d})\theta(\vec{\mathcal{A}}(z)-\vec{\mu}+\vec{d})}{\theta(\vec{\mu})^2 \theta(\vec{\mathcal{A}}(z)+\vec{d})^2}$.

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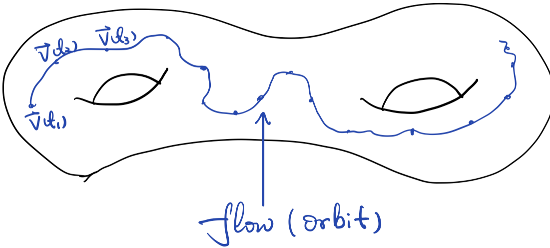
Consider $\mathcal{L}(p, \vec{V}(t))$ for $p \in \mathcal{I}_e := \{a_j, b_j\}_{j=0}^n$

- $\mathcal{L}(p, \vec{V}(t))$ is real-valued and analytic.
- $\mathcal{L}(p, \cdot)$ is periodic on $\mathbb{R}^n/\mathbb{Z}^n$.

$\Longrightarrow$ $\vec{V}(t)$ is a **linear flow (orbit)** on the torus $\mathbb{R}^n/\mathbb{Z}^n$.



Large gap asymptotics: general case

Theorem (X.-Zhang-Zhao, '25)

Let $\mathcal{F}(s\Sigma) := \det(1 - \mathcal{K}^{(\alpha,\beta)}|_{s\Sigma})$ *and* $\Sigma := \cup_{j=0}^{n}(a_j, b_j)$ *be such that* $a_1 < b_1 < \cdots < a_m < 0 < b_m < \ldots < a_n < b_n$ *for some* $0 \leqslant m \leqslant n$*. For* $\alpha > -1/2$ *and* $\beta \in i\mathbb{R}$*, we have, as* $s \to +\infty$*,*

$$\log \mathcal{F}(s\Sigma) = -\gamma_0 s^2 - 2i\mathcal{D}_{\infty,1} s + \log\theta\left(\vec{V}(s)\right) + (\beta^2 - \alpha^2)\log s - \frac{1}{16}\sum_{j=0}^{n}\int_{\hat{s}}^{s}\left(\mathcal{L}\left(a_j, \vec{V}(t)\right) + \mathcal{L}\left(b_j, \vec{V}(t)\right)\right)\frac{\mathrm{d}t}{t} + \check{C}_1 + \mathcal{O}(s^{-1}),$$

where

$$\gamma_0 = -\frac{1}{\pi i}\sum_{j=0}^{n}\int_{a_j}^{b_j}\frac{z\mathrm{p}(z)}{\sqrt{\mathcal{R}(z)}_+}\,\mathrm{d}z \in \mathbb{R},$$

$\mathcal{D}_{\infty,1}$ *is purely imaginary and depends on the parameters* α *and* β*,* $\vec{V}(s) \in \mathbb{R}^n$ *is defined above,* $\hat{s} > 0$ *is a sufficiently large number independent of* s*,* $\mathcal{L}(p, \vec{V}(t))$ *is real for* $p \in \mathcal{I}_e := \{a_j, b_j\}_{j=0}^{n}$ *and* $\check{C}_1$ *is an undetermined constant independent of* s*. Moreover, for* $p \in \mathcal{I}_e$*, as* $s \to +\infty$*, we have*

$$\int_{\hat{s}}^{s}\mathcal{L}\left(p, \vec{V}(t)\right)\frac{\mathrm{d}t}{t} = \hat{\mathcal{L}}_p \log s + o(\log s), \quad \hat{\mathcal{L}}_p := \lim_{T\to+\infty}\frac{1}{T}\int_0^T \mathcal{L}\left(p, \vec{V}(t)\right)\mathrm{d}t.$$

Remarks on the general case

- If $n = 0$, $\Sigma = (-1, 1)$, we have

$$\mathrm{p}(z) = z, \quad \gamma_0 = \frac{1}{2}, \quad \mathcal{D}_{\infty,1} = i\alpha, \quad \mathcal{L}(p, \vec{V}(t)) = \mathcal{L}(z, 0) = 2.$$

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- For general $n > 1$, $\beta = 0$ $\Longrightarrow$ The large gap asymptotics for the type-I Bessel kernel determinant on multiple intervals.

- For general $n > 1$, if $\alpha = \beta = 0$ $\Longrightarrow$ The results is consistent with the Eq. (1.34) in [Deift-Its-Zhou, '97].

Large gap asymptotics: "good Diophantine property" case

Question: Could we improve the asymptotics

$$\int_{\hat{s}}^{s} \mathcal{L}\left(p, \vec{V}(t)\right) \frac{\mathrm{d}t}{t} = \hat{\mathcal{L}}_p \log s + o(\log s), \quad \hat{\mathcal{L}}_p := \lim_{T\to+\infty} \frac{1}{T}\int_0^T \mathcal{L}\left(p, \vec{V}(t)\right) \mathrm{d}t.$$

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Definition (good Diophantine property)

The linear flow

$$(0, +\infty) \ni s \mapsto (V_1(s) \mod 1, \quad V_2(s) \mod 1 \quad \dots \quad V_n(s) \mod 1), \quad V_j(s) = \frac{s}{2\pi}\Omega_j + \frac{1}{2\pi}\operatorname{Im}\zeta_j,$$

has “**good Diophantine properties**” if there exist $\delta_1, \delta_2 > 0$ such that

$$|\vec{m}^{\mathrm{T}}\vec{\Omega}| \geqslant \delta_1 \|\vec{m}\|_2^{-\delta_2} \quad \text{for all } \vec{m} \in \mathbb{Z}^{n\times 1} \text{ with } \vec{m}^{\mathrm{T}}\vec{\Omega} \neq 0,$$

where $\vec{\Omega} := (\Omega_1, \dots, \Omega_n)^{\mathrm{T}}$ and $\|\vec{m}\|_2 = |\vec{m}^{\mathrm{T}}\vec{m}|^{\frac{1}{2}}$.

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Let $\Sigma = \cup_{j=0}^{n}(a_j, b_j)$ be fixed such that $a_0 < b_0 < \cdots < a_m < 0 < b_m < \cdots < a_n < b_n$ for some $0 \leqslant m \leqslant n$ and assume that the good diophantine properties holds. As $s \to +\infty$, one has

$$\int_{\hat{s}}^{s} \mathcal{L}\left(p, \vec{V}(t)\right) \frac{\mathrm{d}t}{t} = \hat{\mathcal{L}}_p \log s + C_p + \mathcal{O}(s^{-1}),$$

where $p \in \mathcal{I}_e$ and C_p is independent of s. Thus, we have, as $s \to +\infty$,

$$\log \mathcal{F}(s\Sigma) = -\gamma_0 s^2 - 2i\mathcal{D}_{\infty,1} s + \log \theta\left(\vec{V}(s)\right) + \left[\beta^2 - \alpha^2 - \frac{1}{16}\sum_{j=0}^{n}(\hat{\mathcal{L}}_{a_j} + \hat{\mathcal{L}}_{b_j})\right] \log s + \check{C}_2 + \mathcal{O}(s^{-1}),$$

where $\check{C}_2 = \check{C}_1 - \frac{1}{16}\sum_{j=0}^{n}(C_{a_j} + C_{b_j})$ is a constant independent of s with $\check{C}_1$ as in the asymptotics of the general case.

Large gap asymptotics: ergodic case

Aim 2: Simplify the $\hat{\mathcal{L}}_p := \lim_{T\to+\infty} \frac{1}{T}\int_0^T \mathcal{L}\left(p, \vec{V}(t)\right)\mathrm{d}t$.

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is ergodic in the n-dimensional torus $\mathbb{R}^n/\mathbb{Z}^n$ if $\{\vec{V}(s) \mod \mathbb{Z}^n\}_{s>0}$ is dense in $\mathbb{R}^n/\mathbb{Z}^n$. Equivalently, the linear flow is ergodic in $\mathbb{R}^n/\mathbb{Z}^n$ if $\{\Omega_j\}_{j=1}^n$ are rationally independent, that is, if there exist $(c_1, c_2, \dots, c_n) \in \mathbb{Z}^n$ such that

$$c_1\Omega_1 + c_2\Omega_2 + \cdots + c_n\Omega_n = 0,$$

then $c_1 = c_2 = \cdots = c_n = 0$.

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Theorem (Birkhoff's ergodic theorem)

The time average exists everywhere, and coincides with the space average if f is continuous (or merely Riemann integrable) and the frequencies Ω_j are independent.

Large gap asymptotics: ergodic case

Applying Birkhoff's ergodic theorem to $\hat{\mathcal{L}}_p := \lim_{T\to+\infty} \frac{1}{T}\int_0^T \mathcal{L}\left(p, \vec{V}(t)\right) \mathrm{d}t$

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Let $\Sigma = \cup_{j=0}^n (a_j, b_j)$ be fixed such that $a_0 < b_0 < \cdots < a_m < 0 < b_m < \cdots < a_n < b_n$ for some $0 \leqslant m \leqslant n$ and assume that $\vec{\Omega}$ satisfies ergodic properties. As $s \to +\infty$, one has

$$\int_{\hat{s}}^{s} \mathcal{L}\left(p, \vec{V}(t)\right) \frac{\mathrm{d}t}{t} = \hat{\mathcal{L}}_p \log s + o(\log s), \quad \hat{\mathcal{L}}_p = \frac{h(p)}{\mathrm{p}(p)} \int_{[0,1)^n} \eta(p; u_1, u_2, \ldots u_n)\, \mathrm{d}u_1 \cdots \mathrm{d}u_n.$$

Thus, we have, as $s \to +\infty$,

$$\log \mathcal{F}(s\Sigma) = -\gamma_0 s^2 - 2i\mathcal{D}_{\infty,1} s + \left[\beta^2 - \alpha^2 - \frac{1}{16}\sum_{j=0}^{n}(\hat{\mathcal{L}}_{a_j} + \hat{\mathcal{L}}_{b_j})\right] \log s + o(\log s),$$

where the constant $\hat{\mathcal{L}}_p$, $p \in \mathcal{I}_e$, is explicitly given by the n-fold integral.

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In the case of $n = 1$, the linear flow satisfies both good Diophantine properties and ergodic properties.

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where C is an undetermined constant independent of s.

Remarks

Let

$$\mathcal{S}_D := \left\{\vec{\Omega} : \ \vec{\Omega} \text{ has ``good Diophantine property''}\right\}, \ \mathcal{S}_E := \left\{\vec{\Omega} : \ \vec{\Omega} \text{ is rationally independent}\right\}.$$

Remarks

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$$\mathcal{S}_D := \left\{ \vec{\Omega} : \ \vec{\Omega} \text{ has "good Diophantine property"} \right\}, \ \mathcal{S}_E := \left\{ \vec{\Omega} : \ \vec{\Omega} \text{ is rationally independent} \right\}.$$

We have some examples:

- Genus $n = 1$ case: $\mathcal{S}_D = \mathcal{S}_E = (0, +\infty)$;
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For $n \geqslant 2$, all of the following cases can and do occur for certain choices of the edges points $\{a_j, b_j\}_{j=0}^{n}$.

$$\vec{\Omega} \notin \mathcal{S}_D \cup \mathcal{S}_E, \quad \vec{\Omega} \in \mathcal{S}_D \setminus \mathcal{S}_E, \quad \vec{\Omega} \in \mathcal{S}_E \setminus \mathcal{S}_D, \quad \vec{\Omega} \in \mathcal{S}_D \cap \mathcal{S}_E.$$

[Deift-Its-Zhou, '97]

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- For the n-fold integral obtained in the ergodic case

$$\hat{\mathcal{L}}_p = \frac{h(p)}{\mathrm{p}(p)} \int_{[0,1)^n} \eta(p; u_1, u_2, \ldots u_n) \, \mathrm{d}u_1 \cdots \mathrm{d}u_n,$$

(a) It's been proved that $\hat{\mathcal{L}}_p \equiv 2$ for $n = 1$.
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- The multiplicative constant $\rightsquigarrow$ challenge.

About the proofs

An RH characterization of the $\mathcal{F}(s\Sigma)$

Rely on an integrable structure of the confluent hypergeometric kernel $K^{(\alpha,\beta)}(x,y)$

$$K^{(\alpha,\beta)}(x,y) = \frac{\vec{f}(x)^{\mathsf{T}}\vec{h}(y)}{x-y} = \frac{\sum_{k=1}^{2} f_k(x)h_k(y)}{x-y}.$$

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- $Y(z) := I - \int_{s\Sigma} \frac{\vec{F}(x)\vec{h}(x)^{\mathsf{T}}}{x-z}\,\mathrm{d}x$ satisfies an 2×2 RH problem, where $\vec{F}(z) = (1-\mathcal{K}^{(\alpha,\beta)}|_{s\Sigma})^{-1}\vec{f}(z)$.

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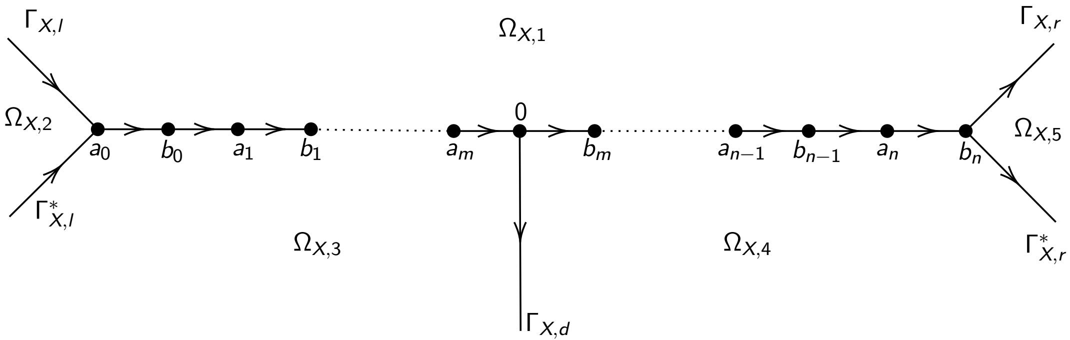
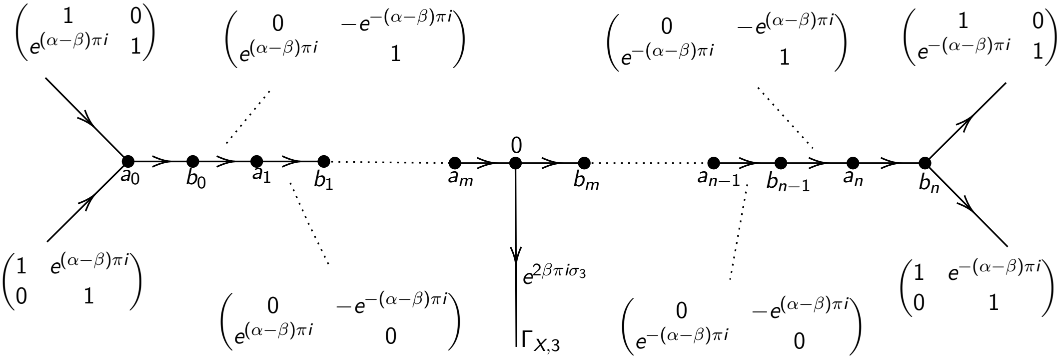
Undressing transform $\rightsquigarrow$ an RH problem for X with constant jumps.

RH problem for X

(a) $X(z)$ is holomorphic for $z \in \mathbb{C} \backslash \Gamma_X$



(b) For $z \in \Gamma_X$, we have $X_+(z) = X_-(z) J_X(z)$.



RH problem for X

(c) As $z \to \infty$, we have

$$X(z) = \left(I + \frac{X_1(s)}{z} + \mathcal{O}\left(z^{-2}\right) \right) z^{-\beta\sigma_3} e^{-isz\sigma_3},$$

where

$$X_1(s) = (2s)^{\beta\sigma_3} \begin{pmatrix} \frac{1}{2s}\left(\Phi_{\mathrm{CH},1}\right)_{11} + \frac{1}{s}(Y_1(s))_{11} & \frac{1}{2s}(\Phi_{\mathrm{CH},1})_{12} + \frac{1}{s}(Y_1(s))_{12} e^{-\pi i\beta} \\ \frac{1}{2s}(\Phi_{\mathrm{CH},1})_{21} + \frac{1}{s}(Y_1(s))_{21} e^{\pi i\beta} & \frac{1}{2s}(\Phi_{\mathrm{CH},1})_{22} + \frac{1}{s}(Y_1(s))_{22} \end{pmatrix} (2s)^{-\beta\sigma_3}$$

with $\Phi_{\mathrm{CH},1} := \lim_{z\to\infty} z(\Phi_{\mathrm{CH}}(z) z^{\beta\sigma_3} e^{\frac{i}{2}z\sigma_3} - I)$.

(d) As $z \to p$ from $\Omega_{X,1}$, $p \in \mathcal{I}_e$, we have $X(z) = \mathcal{O}(\log(z-p))$.

(e) As $z \to 0$ from $\operatorname{Im} z > 0$, we have

$$X(z) = X_0(z) z^{\alpha\sigma_3},$$

where $X_0(z)$ is holomorphic in the neighborhood of 0. The behavior of $X(z)$ as $z \to 0$ is determined by jump conditions.

A “failed” differential identity

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A new differential identity

Proposition (X.-Zhang-Zhao, '25)

We have
$$\partial_s \log \mathcal{F}(s\Sigma) = i\left((X_1(s))_{11} - (X_1(s))_{22}\right) - \frac{\alpha^2 - \beta^2}{s},$$
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Deift-Zhou nonlinear steepest analysis for X

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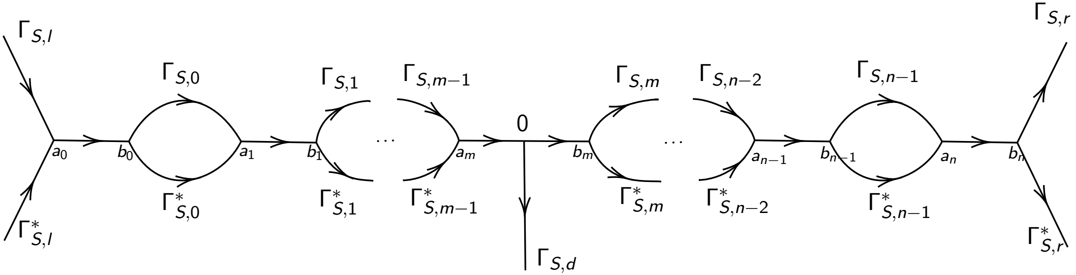
Opening lenses to obtain a solvable RH problem for P



Deift-Zhou nonlinear steepest analysis for X

Global parametrix for $P^{(\infty)}$

- A Szegö function $\mathcal{D}$ to deal with the Fisher-Hartwig singularity.

$$\mathcal{D}(z) := \exp\left\{ \frac{\sqrt{\mathcal{R}(z)}}{2\pi i} \int_{\Sigma} \frac{\mathcal{H}(\xi)}{\xi - z} \mathrm{d}\xi \right\},$$

where

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[Deift-Its-Zhou, '97]

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- Construction of $P^{(\infty)}$: a model RH problem solvable in terms of Riemann-θ function. [Deift-Its-Zhou, '97]
- Some exact formulas related to $P^{(\infty)}$ are required.

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$$= -2s\gamma_0 + i\left((P_1^{(\infty)})_{11} - (P_1^{(\infty)})_{22}\right) + \frac{i}{s}\left((R_1^{(1)})_{11} - (R_1^{(1)})_{22}\right) - \frac{\alpha^2 - \beta^2}{s} + \mathcal{O}(s^{-2}),$$

where $*_1$ is the residue term of $*$ at infinity.

How the two properties help?

Go back to the time average integral

$$I = \frac{1}{T}\int_{\hat{s}}^{T} \mathcal{Y}(\vec{V}(t))\,\mathrm{d}t, \quad \vec{V}(t) = t\vec{\Omega} + \vec{\tilde{\zeta}} \in \mathbb{R}^n/\mathbb{Z}^n.$$

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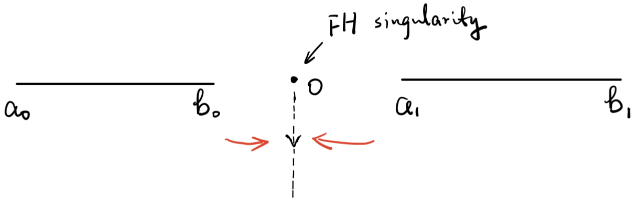
Future work

- Confluent hypergeometric kernel determinant with a FH singularity at the gap and the merging case at the FH singular point.



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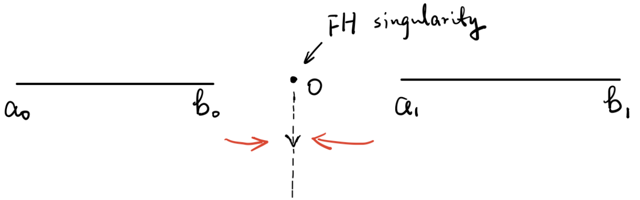

- New differential identity $\rightsquigarrow$ Other integrable kernel determinants for general genus $g > 1$ case?
  (Progress on the Airy kernel determinant)

Thanks for your attention!